Euler's Identity $e^{it} = (\cos t) + i(\sin t)$

Branko Ćurgus

January 28, 2021 22:01

Abstract

In this note I present my variation on the proof of Euler's Identity in which I try to minimize background knowledge that is not presented in the note; there are no citations and I do not use any "well-known" facts. I try to build the proof from "first principles" as much as possible.

1 Preliminary Results

Proposition 1.1. *Let $m \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{m^n}{n!} = 0. \tag{1.1}$$

Proof. The following inequality holds for all $m, n \in \mathbb{N}$

$$\frac{m^n}{n!} \leq \frac{m^m}{(m-1)!}\frac{1}{n}. \tag{1.2}$$

To prove the preceding inequality we notice that it is equivalent to $m^n(m-1)! \leq m^m(n-1)!$. The inequality is trivial if $m = n$. Assume $m > n$. Then $n \cdots (m-1) \leq m^{m-n}$, since the both sides have the same number of factors and the factors on the left-hand side are smaller. Multiplying both sides by $m^n(n-1)!$ yields the desired inequality. Assume $m < n$. Then $m^{n-m} \leq m \cdots (n-1)$, since the both sides have the same number of factors and the factors on the left-hand side are smaller. Multiplying both sides by $m^m(m-1)!$ yields the desired inequality. It follows from (1.2) that for arbitrary $\epsilon > 0$ and $n > (m^m)/(\epsilon(m-1)!)$ we have $(m^n)/(n!) < \epsilon$. Hence, the limit in (1.1) is proved using the definition of limit. □

Theorem 1.2. *Let $r \in \mathbb{R}_+$ and let $I = [-r, r]$ or $I = \mathbb{R}$. Let $g : I \to \mathbb{R}$ be a continuous function. Assume that there exists $M \in \mathbb{R}_+$ and $m \in \{0\} \cup \mathbb{N}$ such that for all $x \in I$ we have*

$$\big|g(x)\big| \leq M|x|^m. \tag{1.3}$$

Then for all $x \in I$ we have

$$\left|\int_0^x g(t)dt\right| \leq \frac{M}{m+1}|x|^{m+1}. \tag{1.4}$$

Proof. Assume that there exists $M \in \mathbb{R}_+$ and $m \in \{0\} \cup \mathbb{N}$ such that (1.3) holds for all $x \in I$. From the definition of the absolute value function it follows that (1.3) is equivalent to

$$-M|t|^m \leq g(t) \leq M|t|^m \tag{1.5}$$

for all $t \in I$. **Case 1.** Assume $x \in I$ and $x > 0$. Then for every $t \in [0, x]$ we have that (1.5) holds and we can drop the absolute value sign. By the monotonicity property of the definite integral we get

$$-M\int_0^x t^m dt \leq \int_0^x g(t)dt \leq M\int_0^x t^m dt.$$

Consequently

$$-\frac{M}{m+1}x^{m+1}dt \leq \int_0^x g(t)dt \leq \frac{M}{m+1}x^{m+1},$$

which is equivalent to

$$\left|\int_0^x g(t)dt\right| \leq \frac{M}{m+1}|x|^{m+1}. \tag{1.6}$$

Case 2. Assume $x \in I$ and $x < 0$. Then for every $t \in [x, 0]$ we have that (1.5) holds and we can replace $|t|$ by $(-t)$. By the monotonicity property of the definite integral we get

$$-M\int_x^0 (-t)^m dt \leq \int_x^0 g(t)dt \leq M\int_x^0 (-t)^m dt$$

and consequently

$$-\frac{M}{m+1}(-x)^{m+1}dt \le \int_x^0 g(t)dt \le \frac{M}{m+1}(-x)^{m+1}.$$

Multiplying the last expression by -1 and replacing $(-x)$ by $|x|$ we obtain

$$-\frac{M}{m+1}|x|^{m+1}dt \le \int_0^x g(t)dt \le \frac{M}{m+1}|x|^{m+1},$$

which is equivalent to

$$\left|\int_0^x g(t)dt\right| \le \frac{M}{m+1}|x|^{m+1}.$$

The preceding inequality and (1.6) prove that (1.4) holds for all $x \in I$. □

Next we define three operations on functions inspired by the anti-derivative from the previous theorem. For a continuous function $g : \mathbb{R} \to \mathbb{R}$ set

$$(\mathcal{I}g)(x) = \int_0^x g(t)dt,$$
$$(\mathcal{J}g)(x) = 1 - \int_0^x g(t)dt,$$
$$(\mathcal{K}g)(x) = 1 + \int_0^x g(t)dt.$$

Corollary 1.3. *Let $r \in \mathbb{R}_+$ and let $I = [-r, r]$ or $I = \mathbb{R}$. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be continuous functions. Assume that there exists $M \in \mathbb{R}_+$ and $m \in \{0\} \cup \mathbb{N}$ such that for all $x \in I$ we have*

$$\big|f(x) - g(x)\big| \le M|x|^m. \tag{1.7}$$

Then for all $x \in I$ we have

$$\big|(\mathcal{I}f)(x) - (\mathcal{I}g)(x)\big| \le \frac{M}{m+1}|x|^{m+1}, \tag{1.8}$$
$$\big|(\mathcal{J}f)(x) - (\mathcal{J}g)(x)\big| \le \frac{M}{m+1}|x|^{m+1}, \tag{1.9}$$
$$\big|(\mathcal{K}f)(x) - (\mathcal{K}g)(x)\big| \le \frac{M}{m+1}|x|^{m+1}. \tag{1.10}$$

Proof. To prove (1.8) we calculate

$$(\mathcal{I}f)(x) - (\mathcal{I}g)(x) = \int_0^x f(t)dt - \int_0^x g(t)dt = \int_0^x \big(f(t) - g(t)\big)dt$$

and apply Theorem 1.2 to deduce (1.8) from (1.7). To prove (1.9) we calculate

$$\big|(\mathcal{J}f)(x) - (\mathcal{J}g)(x)\big| = \left|1 - \int_0^x f(t)dt - 1 + \int_0^x g(t)dt\right| = \left|\int_0^x \big(f(t) - g(t)\big)dt\right|$$

and apply Theorem 1.2 to deduce (1.9) from (1.7). To prove (1.10) we calculate

$$\big|(\mathcal{K}f)(x) - (\mathcal{K}g)(x)\big| = \left|1 + \int_0^x f(t)dt - 1 - \int_0^x g(t)dt\right| = \left|\int_0^x \big(f(t) - g(t)\big)dt\right|$$

and apply Theorem 1.2 to deduce (1.10) from (1.7). □

2 The Exponential Function

Theorem 2.1. *Let $r \in \mathbb{R}_+$ be arbitrary. Then for all $n \in \mathbb{N}$ and all $x \in [-r, r]$ we have*

$$\left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| \le e^r \frac{|x|^{n+1}}{(n+1)!} \tag{2.1}$$

Proof. **Part 1.** First we establish a pattern how repeated application of the operation $\mathcal{K}$ starting with the constant 1 creates a sequence of polynomials. We start by applying $\mathcal{K}$ to 1, then we apply $\mathcal{K}$ to the result $(\mathcal{K}1)(x)$ and so on. We obtain the following sequence of polynomials

$$(\mathcal{K}1)(x) = 1 + \int_0^x 1dt = 1 + x, \tag{2.2}$$

$$(\mathcal{K}^2 1)(x) = 1 + \int_0^x (1+t)dt = 1 + x + \frac{x^2}{2}, \tag{2.3}$$

$$(\mathcal{K}^3 1)(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right)dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!},$$

$$(\mathcal{K}^4 1)(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2} + \frac{t^3}{3!}\right)dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

In general, for all $n \in \mathbb{N}$ we have

$$(\mathcal{K}^n 1)(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{2.4}$$

Part 2. Let $r > 0$ be arbitrary. Then for all $x \in [-r, r]$ we have

$$\left| e^x \right| \le e^r.$$

Applying Theorem 1.2 to the preceding inequality yields

$$\left| e^x - 1 \right| \le e^r |x| \tag{2.5}$$

for all $x \in [-r, r]$. The preceding inequality proves (2.1) for $n = 0$.

Part 3. In this part of the proof we use the fact that the operation $\mathcal{K}$ does not change the exponential function. That is,

$$(\mathcal{K}\exp)(x) = 1 + \int_0^x e^t dt = 1 + e^x - 1 = e^x = \exp(x).$$

Step 1. Apply $\mathcal{K}$ to both functions $\exp x$ and 1 in (2.5) and use (1.10) to conclude

$$\left| (\exp x) - (\mathcal{K}1)(x) \right| \le \frac{e^r}{2!} |x|^2 \tag{2.6}$$

for all $x \in [-r, r]$. Since (2.2) holds, we see that (2.6) proves (2.1) for $n = 1$.

Step 2. Apply $\mathcal{K}$ to both functions $\exp x$ and $(\mathcal{K}1)(x)$ in (2.6) and use (1.10) to conclude

$$\left| (\exp x) - (\mathcal{K}^2 1)(x) \right| \le \frac{e^r}{3!} |x|^3 \tag{2.7}$$

for all $x \in [-r, r]$. Since (2.3) holds, we see that (2.6) proves (2.1) for $n = 2$.

Repeating these steps for a total of n times we deduce

$$\left| (\exp x) - (\mathcal{K}^n 1)(x) \right| \le \frac{e^r}{(n+1)!} |x|^{n+1}.$$

Since (2.4) holds, the preceding inequality proves (2.1). ∎

Corollary 2.2. *For all $x \in \mathbb{R}$ we have*

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k. \tag{2.8}$$

Proof. □

3 The Cosine and Sine Functions

In this section we utilize the reasoning very similar to the reasoning from Section 2 to deduce similar conclusions for the cosine and sine function. Since we are dealing with two functions, instead of one operation $\mathcal{K}$ used in Section 2, here we use two operations $\mathcal{I}$ and $\mathcal{J}$ and apply them successively.

Theorem 3.1. *For all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$ we have*

$$\left| \cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} \right| \leq \frac{|x|^{2n+1}}{(2n+1)!} \tag{3.1}$$

and

$$\left| \sin x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right| \leq \frac{|x|^{2n+2}}{(2n+2)!} \tag{3.2}$$

Proof. **Part 1.** First we establish a pattern how repeated application of the operations $\mathcal{I}$ and $\mathcal{J}$ starting with the constant 1 creates a sequence of polynomials. We start by applying $\mathcal{I}$ to 1, then we apply $\mathcal{J}$ to the result and so on. We obtain

$$(\mathcal{I}1)(x) = \int_0^x 1 dt = x, \tag{3.3}$$

$$((\mathcal{J} \circ \mathcal{I})1)(x) = 1 - \int_0^x t dt = 1 - \frac{x^2}{2}, \tag{3.4}$$

$$(\mathcal{I} \circ (\mathcal{J} \circ \mathcal{I})1)(x) = \int_0^x \left(1 - \frac{t^2}{2}\right) dt = x - \frac{x^3}{3!}, \tag{3.5}$$

$$((\mathcal{J} \circ \mathcal{I})^2 1)(x) = 1 - \int_0^x \left(t - \frac{t^3}{3!}\right) dt = 1 - \frac{x^2}{2} + \frac{x^4}{4!}, \tag{3.6}$$

$$(\mathcal{I} \circ (\mathcal{J} \circ \mathcal{I})^2 1)(x) = \int_0^x \left(1 - \frac{t^2}{2} + \frac{t^4}{4!}\right) dt = x - \frac{x^3}{3!} + \frac{x^5}{5!}, \tag{3.7}$$

$$((\mathcal{J} \circ \mathcal{I})^3 1)(x) = 1 - \int_0^x \left(t - \frac{t^3}{3!} + \frac{t^5}{5!}\right) dt = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!},$$

$$(\mathcal{I} \circ (\mathcal{J} \circ \mathcal{I})^3 1)(x) = \int_0^x \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!}\right) dt = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

In general, for all $n \in \mathbb{N}$ we have

$$((\mathcal{J} \circ \mathcal{I})^n 1)(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n}{(2n)!} x^{2n} = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} \tag{3.8}$$

and

$$\mathcal{I} \circ ((\mathcal{J} \circ \mathcal{I})^n 1)(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \tag{3.9}$$

Part 2. The basic property of the sine function is that for all $x \in \mathbb{R}$ we have

$$\big|-\sin x\big| \le 1.$$

Here $g(x) = -\sin x$, $M = 1$ and $m = 0$ in (1.3). Theorem 1.2 applied to the preceding inequality yields

$$\big|(\cos x) - 1\big| \le |x| \tag{3.10}$$

for all $x \in \mathbb{R}$. Now we apply Theorem 1.2 again to get

$$\big|(\sin x) - x\big| \le \frac{1}{2}|x|^2 \tag{3.11}$$

for all $x \in \mathbb{R}$. Inequalities (3.10) and (3.11) prove inequalities (3.1) and (3.2) for $n = 0$.
Part 3. In this part of the proof we use the following properties of the operations $\mathcal{I}$ and $\mathcal{J}$

$$(\mathcal{J}\sin)(x) = \cos(x) \quad \text{and} \quad (\mathcal{I}\cos)(x) = \sin(x).$$

Step 1. We apply $\mathcal{J}$ to both functions $\sin x$ and $x = (\mathcal{I}1)(x)$ in the difference in (3.11) and use (1.9) to conclude

$$\big|(\cos x) - ((\mathcal{J}\circ\mathcal{I})1)(x)\big| \le \frac{1}{3!}|x|^3. \tag{3.12}$$

Further, we apply $\mathcal{I}$ to the preceding inequality and use (1.8) to obtain

$$\big|(\sin x) - (\mathcal{I}\circ(\mathcal{J}\circ\mathcal{I})1)(x)\big| \le \frac{1}{4!}|x|^4. \tag{3.13}$$

Using the equalities established in Part 1 of this proof we see that (3.12) and (3.13) prove (3.1) and (3.2) for $n = 1$.
Step 2. We apply $\mathcal{J}$ to both functions in the difference in (3.13) and use (1.9) to conclude

$$\big|(\cos x) - ((\mathcal{J}\circ\mathcal{I})^2 1)(x)\big| \le \frac{1}{5!}|x|^5. \tag{3.14}$$

Further, we apply $\mathcal{I}$ to the preceding inequality and use (1.8) to obtain

$$\big|(\sin x) - (\mathcal{I}\circ(\mathcal{J}\circ\mathcal{I})^2 1)(x)\big| \le \frac{1}{6!}|x|^6. \tag{3.15}$$

Using the equalities established in Part 1 of this proof we see that (3.14) and (3.15) prove (3.1) and (3.2) for $n = 2$.
Step n. Repeating these steps for a total of n times we obtain

$$\big|(\cos x) - ((\mathcal{J}\circ\mathcal{I})^n 1)(x)\big| \le \frac{1}{(2n+1)!}|x|^{2n+1}.$$

and

$$\big|(\sin x) - (\mathcal{I}\circ(\mathcal{J}\circ\mathcal{I})^n 1)(x)\big| \le \frac{1}{(2n+2)!}|x|^{2n+2}.$$

With the equalities established in Part 1 of this proof, the preceding two inequalities prove (3.1) and (3.2). ☐

Corollary 3.2. *For all $x \in \mathbb{R}$ we have*

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad \textit{and} \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}. \tag{3.16}$$

4 Euler's Identity

In Corollary 2.2 we proved that for all $x \in \mathbb{R}$ we have

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n. \tag{4.1}$$

A remarkable feature of this equality is that the value e^x of the exponential function is expressed as a sum of the nonnegative powers of x; nonnegative powers of x being the simplest functions of x.

The series representation of e^x in (4.1) can be used to understand the exponentiation with imaginary numbers.

Recall that the imaginary unit i is defined as a complex number whose square is -1. That is $i^2 = -1$. A general complex number z is commonly represented as a sum $z = a + ib$, where a and b are real numbers. In this representation a is called the real part of z and b is called the imaginary part of z. Doing calculations with complex numbers, the objective is always to represent a complex number as a sum of its real and imaginary part multiplied by i. For example, when multiplying two complex numbers $z = a + ib$ and $w = c + id$ the product is

$$zw = (a + ib)(c + id) = ac + iad + ibc + i^2 bd = (ac - bd) + i(ad + bc).$$

Thus, the real part of the product $zw = (a + ib)(c + id)$ is the real number $ac - bd$ while the imaginary part of the product is $ad + bc$.

So we ask:

> For a real number t, what is the real part and what is the imaginary part of the complex number e^{it}?

To answer this question we resort to the series representation (4.1), we replace x by it and define

$$e^{it} = \sum_{n=0}^{\infty} \frac{1}{n!} (it)^n. \tag{4.2}$$

Now we do algebra with the infinite series with complex numbers $(it)^k$ with $k \in \{0\} \cup \mathbb{N}$. Since the multiplication of complex numbers works the same as with real numbers we have

$$(it)^n = i^n t^n \quad \text{where} \quad n \in \{0\} \cup \mathbb{N}.$$

Now we need to understand the complex numbers i^n with $n \in \{0\} \cup \mathbb{N}$.

$$\begin{array}{llll} i^0 = 1, & i^1 = i, & i^2 = -1, & i^3 = -i, \\ i^4 = 1 & i^5 = i & i^6 = -1 & i^7 = -i \\ i^8 = 1 & i^9 = i & i^{10} = -1 & i^{11} = -i. \end{array}$$

We distinguish two cases: n is even, that is $n = 2k$ with $k \in \{0\} \cup \mathbb{N}$ and n is odd, that is $n = 2k+1$ with $k \in \{0\} \cup \mathbb{N}$. For n even we have

$$i^n = i^{2k} = \left(i^2\right)^k = (-1)^k. \tag{4.3}$$

For n odd we have

$$i^n = i^{2k+1} = ii^{2k} = i\big(i^2\big)^k = i(-1)^k. \tag{4.4}$$

Now we are ready to further expand (4.2):

$$\begin{aligned}
e^{it} &= \sum_{n=0}^{\infty} \frac{1}{n!}(it)^n \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!}t^n && \text{algebra } (it)^n = i^n t^n \\
&= \sum_{n \text{ is even}} \frac{i^n}{n!}t^n + \sum_{n \text{ is odd}} \frac{i^n}{n!}t^n && \text{separate even and odd} \\
&= \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!}t^{2k} + \sum_{k=0}^{\infty} \frac{i^{2k+1}}{(2k-1)!}t^{2k-1} && n = 2k \text{ for even and } n = 2k+1 \text{ for odd} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}t^{2k} + i\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k-1)!}t^{2k-1} && \text{see (4.3) and (4.4)} \\
&= (\cos t) + i(\sin t). && \text{see (3.16)}
\end{aligned}$$

Hence, we proved Euler's identity

$$\boxed{e^{it} = (\cos t) + i(\sin t) \quad \text{for all} \quad t \in \mathbb{R}}$$