THE FUNDAMENTAL THEOREM OF ALGEBRA

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In this note I present a proof of the Fundamental Theorem of Algebra which is based on the algebra of complex numbers, Euler's formula, continuity of polynomials and the extreme value theorem for continuous functions. The main argument in this note is similar to [2]. In [3] the reader can find another proof and more references to different proofs of the Fundamental Theorem of Algebra. The entire book [1] is devoted to different proofs of the Fundamental Theorem of Algebra.

Let $n \in \mathbb{N}$ and let $a_0, a_1, \ldots, a_{n-1}, a_n$ be complex numbers with $a_n \neq 0$. Let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$$

be a polynomial of degree n. In this note z is always a complex number.

Lemma 1. *Set*

$$R_p = \max\left\{1, \frac{2}{|a_n|} \sum_{j=0}^{n-1} |a_j| \right\}.$$

For every $z \in \mathbb{C}$ the following two implications hold:

$$|z| \geq R_p \quad \Rightarrow \quad \frac{1}{2}|a_n||z|^n \leq |p(z)| \leq \frac{3}{2}|a_n||z|^n, \tag{1}$$

$$|z| \geq R_p \quad \Rightarrow \quad |p(0)| \leq |p(z)|. \tag{2}$$

Proof. Assume that $|z| \geq R_p$. Since $|z| \geq \frac{2}{|a_n|} \sum_{j=0}^{n-1} |a_j|$, we have $\frac{1}{|z|} \sum_{j=0}^{n-1} |a_j| \leq \frac{|a_n|}{2}$. The triangle inequality, the assumption $|z| \geq 1$ and the preceding inequality yield

$$\left| \sum_{j=0}^{n-1} \frac{a_j}{z^{n-j}} \right| \leq \sum_{j=0}^{n-1} \frac{|a_j|}{|z|^{n-j}} \leq \frac{1}{|z|} \sum_{j=0}^{n-1} |a_j| \leq \frac{|a_n|}{2}. \tag{3}$$

Further, the triangle inequality and (3) yield

$$\frac{1}{2}|a_n| \leq |a_n| - \left| \sum_{j=0}^{n-1} \frac{a_j}{z^{n-j}} \right| \leq \left| \frac{p(z)}{z^n} \right| \leq |a_n| + \left| \sum_{j=0}^{n-1} \frac{a_j}{z^{n-j}} \right| \leq \frac{3}{2}|a_n|.$$

Multiplying the first, the third and the fifth term in the preceding relation by $|z|^n > 0$ completes the proof of (1).

Date: February 5, 2017 at 14:23.

To prove (2) assume again that $|z| \geq R_p$. Since $1 \leq |z| \leq |z|^n$, the assumption $2|a_0|/|a_n| \leq 1$ yields $2|a_0|/|a_n| \leq |z|^n$. On the other hand, if $2|a_0|/|a_n| > 1$, then

$$1 < \sqrt[n]{\frac{2}{|a_n|}|a_0|} \leq \frac{2}{|a_n|}|a_0| \leq \frac{2}{|a_n|}\sum_{j=0}^{n-1}|a_j| \leq R_p \leq |z|.$$

Consequently, again, $2|a_0|/|a_n| \leq |z|^n$. Hence $2|a_0|/|a_n| \leq |z|^n$ whenever $|z| \geq R_p$. The preceding fact and (1) yield

$$|p(z)| \geq \frac{1}{2}|a_n||z|^n \geq \frac{1}{2}|a_n|\frac{2|a_0|}{|a_n|} = |a_0|.$$

This proves (2). □

In the following lemma for $r > 0$ we set

$$\mathbb{D}(r) = \bigl\{z \in \mathbb{C} \,|\, |z| \leq r\bigr\}.$$

Proposition 2. *There exists $c \in \mathbb{C}$ such that $|p(c)| \leq |p(z)|$ for all $z \in \mathbb{C}$.*

Proof. Recall (2) in Lemma 1:

$$\forall\, z \notin \mathbb{D}\bigl(R_p\bigr) \qquad |p(0)| \leq |p(z)|. \tag{4}$$

By the extreme values theorem the continuous function $z \mapsto |p(z)|$ has a minimum on the closed disk $\mathbb{D}\bigl(R_p\bigr)$. That is, there exists $c \in \mathbb{D}\bigl(R_p\bigr)$ such that

$$\forall\, z \in \mathbb{D}\bigl(R_p\bigr) \qquad |p(c)| \leq |p(z)|. \tag{5}$$

Since $0 \in \mathbb{D}\bigl(R_p\bigr)$ we have $|p(c)| \leq |p(0)|$. Therefore (4) and (5) yield

$$\forall\, z \in \mathbb{C} \qquad |p(c)| \leq |p(z)|,$$

completing the proof of the proposition. □

Theorem 3. *There exists $c \in \mathbb{C}$ such that $p(c) = 0$.*

Proof. By Proposition 2 there exists $c \in \mathbb{C}$ such that for all $z \in \mathbb{C}$ we have $|p(c)| \leq |p(z)|$. By considering the polynomial $p(z + c)$ instead of $p(z)$ we can assume that $c = 0$. That is, we can assume

$$\forall\, z \in \mathbb{C} \qquad |p(0)| \leq |p(z)|. \tag{6}$$

We will prove that (6) implies $p(0) = 0$.

Set

$$k = \min\bigl\{j \in \{1, \ldots, n\} \,|\, a_j \neq 0\bigr\}.$$

Further set

$$p(z) = p(0) + z^k q(z),$$

where, by the definition of k, $q(0) = a_k \neq 0$ is a polynomial of degree $n - k$.

Next calculate

$$|p(z)|^2 - |p(0)|^2 = |p(0) + z^k q(z)|^2 - |p(0)|^2 = 2\,\mathrm{Re}\bigl(\overline{p(0)}q(z)z^k\bigr) + |z|^{2k}|q(z)|^2$$

and rewrite (6) as

$$\forall\, z \in \mathbb{C} \qquad 2\operatorname{Re}\big(\overline{p(0)}q(z)z^k\big) + |z|^{2k}|q(z)|^2 \geq 0.$$

Setting $z = re^{it}$ with $t \in \mathbb{R}$, $r > 0$ and dividing by $2r^k > 0$ yields

$$\forall\, t \in \mathbb{R} \;\; \forall\, r > 0 \qquad \operatorname{Re}\big(\overline{p(0)}q(re^{it})e^{ikt}\big) + \frac{1}{2}r^k|q(re^{it})|^2 \geq 0.$$

Taking the limit as $r \to 0$ in the last expression we obtain

$$\forall\, t \in \mathbb{R} \qquad \operatorname{Re}\big(\overline{p(0)}q(0)e^{ikt}\big) \geq 0.$$

With $\theta = \operatorname{Arg}\big(\overline{p(0)}q(0)\big)$ the last inequality reads

$$\forall\, t \in \mathbb{R} \qquad \big|\overline{p(0)}q(0)\big| \cos(\theta + kt) \geq 0.$$

Setting $t = (\pi - \theta)/k \in \mathbb{R}$ yields

$$-\big|\overline{p(0)}q(0)\big| \geq 0.$$

Since always $\big|\overline{p(0)}q(0)\big| \geq 0$ we have $\big|\overline{p(0)}q(0)\big| = 0$. As by the definition of q we have $q(0) \neq 0$, this proves $p(0) = 0$. □

References

[1] Fine, Benjamin; Rosenberger, Gerhard: The fundamental theorem of algebra. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1997.

[2] de Oliveira, Oswaldo Rio Branco: The fundamental theorem of algebra: an elementary and direct proof. Math. Intelligencer 33 (2011), no. 2, 1–2.

[3] de Oliveira, Oswaldo Rio Branco: The fundamental theorem of algebra: from the four basic operations. Amer. Math. Monthly 119 (2012), no. 9, 753–758.