$\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable

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The symbol $\mathbb{Z}^+$ denotes the set of positive integers.

Definition 1. A set A is *countably infinite* if there exists a bijection $f : \mathbb{Z}^+ \to A$.

The goal of this note is to provide a rigorous proof that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countably infinite. That is to provide a specific bijection from $\mathbb{Z}^+$ to $\mathbb{Z}^+ \times \mathbb{Z}^+$. Below such a bijection is called B.

First some preliminaries. Recall that the sequence of triangular numbers is given by

$$T_n = \frac{n(n+1)}{2}, \qquad n \in \mathbb{Z}^+.$$

It is convenient to also define $T_0 = 0$.

Exercise 2. Prove that for every $n \in \mathbb{Z}^+$ we have $n \le T_n$.

Solution. Let $n \in \mathbb{Z}^+$ be arbitrary. Multiplying each side of the inequality $1 \le n$ by $n > 0$ we get $n \le n^2$. Adding n to each side of the last inequality yields $2n \le n^2 + n$, that is, $2n \le n(n+1)$. Dividing by 2 yields $n \le T_n$. $\square$

To get an idea how triangular numbers are spaced among positive integers we present the following table. The **triangular numbers** are in bold face.

	T_1		T_2			T_3				T_4					T_5						T_6		
n	**1**	2	**3**	4	5	**6**	7	8	9	**10**	11	12	13	14	**15**	16	17	18	19	20	**21**	22	23
R_n	1	2	2	3	3	3	4	4	4	4	5	5	5	5	5	6	6	6	6	6	6	7	7

The table above indicates that the following sequence

$$R_1 = 1,\ R_2 = 2,\ R_3 = 2,\ R_4 = 3,\ R_5 = 3,\ R_6 = 3,\ R_7 = 4,\ R_8 = 4,\ R_9 = 4,\ R_{10} = 4,\ R_{11} = 5,\ R_{12} = 5, \ldots$$

is closely related to the sequence of triangular numbers. For a given $n \in \mathbb{Z}^+$ R_n is the index of the smallest triangular number which is larger or equal than n. Formally, we define the sequence $R : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$R_n = \min\bigl\{k \in \mathbb{Z}^+ \, : \, n \le T_k \bigr\}, \qquad n \in \mathbb{Z}^+.$$

The above definition uses the concept of minimum. To make this definition rigorous, we need to prove that the above minimum exists. By Exercise 2 for arbitrary $n \in \mathbb{Z}^+$ we have $n \le T_n$. Therefore $n \in \bigl\{k \in \mathbb{Z}^+ : n \le T_k \bigr\}$; that is, the set $\bigl\{k \in \mathbb{Z}^+ : n \le T_k \bigr\}$ is a nonempty set of positive integers. By the well ordering axiom this set has a minimum. This justifies the definition of R_n.

By the definition of minimum, R_n belongs to the set $\bigl\{k \in \mathbb{Z}^+ : n \le T_k \bigr\}$. Therefore $n \le T_{R_n}$. Also, by the definition of minimum $R_n - 1$ does not belong to the set $\bigl\{k \in \mathbb{Z}^+ : n \le T_k \bigr\}$. Therefore $T_{R_n - 1} < n$. Thus, for every $n \in \mathbb{Z}^+$ we have

$$T_{R_n - 1} < n \le T_{R_n}. \tag{1}$$

In other words, for an arbitrary $n \in \mathbb{Z}^+$, the integer R_n provides the index of the smallest triangular number which is $\ge n$. Notice that (1) also claims that n is larger than the triangular number with index $R_n - 1$.

Remark 3. There are several other formulas for the sequence R. For example, for $n \in \mathbb{Z}^+$,

$$R_n = \left\lfloor \frac{1}{2} + \sqrt{2n} \right\rfloor = \left\lceil -\frac{1}{2} + \sqrt{2n} \right\rceil .$$

Here $\lfloor \cdot \rfloor$ is the floor function, $\lceil \cdot \rceil$ is the ceiling function and $\sqrt{\cdot}$ is the square root function.

Recall that

$$\mathbb{Z}^+ \times \mathbb{Z}^+ := \bigl\{ (s,t) \,:\, s,t \in \mathbb{Z}^+ \bigr\}.$$

The set $\mathbb{Z}^+ \times \mathbb{Z}^+$ is illustrated by the following infinite table:

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$\dots$
$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$\dots$
$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$\dots$
$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$\dots$
$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Rearranging the pairs we can enumerate them with positive integers. This enumeration is demonstrated in the table below. Each pair is enumerated by a positive integer placed in a small circle. Usually the table below is considered to be a proof of the countability of $\mathbb{Z}^+ \times \mathbb{Z}^+$.

① $(1,1)$	② $(1,2)$	④ $(1,3)$	⑦ $(1,4)$	⑪ $(1,5)$	⑯ $(1,6)$	$\dots$
	③ $(2,1)$	⑤ $(2,2)$	⑧ $(2,3)$	⑫ $(2,4)$	⑰ $(2,5)$	$\dots$
		⑥ $(3,1)$	⑨ $(3,2)$	⑬ $(3,3)$	⑱ $(3,4)$	$\dots$
			⑩ $(4,1)$	⑭ $(4,2)$	⑲ $(4,3)$	$\dots$
				⑮ $(5,1)$	⑳ $(5,2)$	$\dots$
					㉑ $(6,1)$	$\dots$
						$\ddots$

Table 1: Labeled $\mathbb{Z}^+ \times \mathbb{Z}^+$

If one accepts the above enumeration table as a proof, then one would never know which pair is associated with the positive integer 321, or, which circled positive integer is used to enumerate the pair $(21,5)$. Furthermore, the enumeration table above poses an interesting challenge: find a *formula* for the function $B : \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ which is indicated by the table. Since we expect such B to be a bijection, we also need to find a formula for its inverse, call it $A : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, such that

$$B\bigl(A(s,t)\bigr) = (s,t) \;\; \forall\, (s,t) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \quad \text{and} \quad A\bigl(B(n)\bigr) = n \;\; \forall\, n \in \mathbb{Z}^+. \tag{2}$$

Two identities in (2) are equivalent to the statement: B is a bijection.

Notice that the circled labels along the diagonal in Table 1 are triangular numbers. The pattern is clear: the label for $(s,1)$ is the triangular number T_s. The sum of the entries of each pair in the same column as $(s,1)$ is $s+1$. The labels decrease as we climb up the column, that is as t increases. This gives us the function $A:\mathbb{Z}^+\times\mathbb{Z}^+\to\mathbb{Z}^+$:

$$A(s,t)=\frac{(s+t-1)(s+t)}{2}+1-t, \qquad s,t\in\mathbb{Z}^+.$$

Next we have to figure out a pair associated with $n\in\mathbb{Z}^+$. As we have noticed before triangular numbers play an important role in the labeling. As we can see from Table 1 the numbers s and t are related to how far n is from the previous and the following triangular number. We already know from (1) that

$$\frac{(R_n-1)R_n}{2}<n\leq\frac{R_n(R_n+1)}{2}.$$

Now it is not difficult to see from Table 1 that $B:\mathbb{Z}^+\to\mathbb{Z}^+\times\mathbb{Z}^+$ is given by

$$B(n)=\Bigg(\underbrace{n-\frac{(R_n-1)R_n}{2}}_{\substack{\text{distance to the preced-}\\\text{ing triangular number}}},\ \underbrace{\frac{R_n(R_n+1)}{2}-n}_{\substack{\text{distance to the follow-}\\\text{ing triangular number}}}+1\Bigg), \qquad n\in\mathbb{Z}^+.$$

By the definition of triangular numbers the formulas for A and B can be written as

$$\begin{aligned} A(s,t)&=T_{(s+t-1)}+1-t, & s,t\in\mathbb{Z}^+,\\ B(n)&=\Big(n-T_{(R_n-1)}\,,\,T_{R_n}-n+1\Big), & n\in\mathbb{Z}^+.\end{aligned}$$

Let $s,t\in\mathbb{Z}^+$. We evaluate $R_{(T_{(s+t-1)}+1-t)}$ first. Since $0<s$ and $0\leq t-1$ we have

$$T_{(s+t-2)}=T_{(s+t-1)}-(s+t-1)<T_{(s+t-1)}+1-t\leq T(s+t-1).$$

In the first equality above we used the identity

$$T_k=T_{(k-1)}+k, \tag{3}$$

which follows from

$$T_k-T_{(k-1)}=\frac{k(k+1)}{2}-\frac{(k-1)k}{2}=\frac{k^2+k-k^2+k}{2}=k.$$

Hence, the integer $T_{(s+t-1)}+1-t$ is squeezed between two consecutive triangular numbers:

$$T_{(s+t-2)}<T_{(s+t-2)}+s=T_{(s+t-2)}+s+t-1+1-t=T_{(s+t-1)}+1-t\leq T_{(s+t-1)},$$

so, by (1),

$$R_{\left(T_{(s+t-1)}+1-t\right)}=s+t-1.$$

We have thus calculated that

$$R_{A(s,t)}=s+t-1.$$

Next we use the last identity, the definitions of A and B and (3) to calculate

$$\begin{aligned} B\big(A(s,t)\big)&=\Big(A(s,t)-\frac{\big(R_{A(s,t)}-1\big)R_{A(s,t)}}{2}\Big)\,,\,\frac{R_{A(s,t)}\big(R_{A(s,t)}+1\big)}{2}-A(s,t)+1\Big)\\ &=\Big(A(s,t)-\frac{(s+t-1-1)(s+t-1)}{2}\Big)\,,\,\frac{(s+t-1)(s+t-1+1)}{2}-A(s,t)+1\Big)\\ &=\Big(A(s,t)-T_{(s+t-2)}\,,\,T_{(s+t-1)}-A(s,t)+1\Big)\\ &=\Big(T_{(s+t-1)}+1-t-T_{(s+t-2)}\,,\,T_{(s+t-1)}-\big(T_{(s+t-1)}+1-t\big)+1\Big)\\ &=\big(s+t-1+1-t\,,\,t\big)\\ &=(s,t).\end{aligned}$$

This proves $B\big(A(s,t)\big) = (s,t)$ for all $s, t \in \mathbb{Z}^+$.

Let $n \in \mathbb{Z}^+$ be arbitrary. Before proceeding with the proof $A\big(B(n)\big) = n$, notice that by (3) the sum of entries in the pair $B(n)$ is

$$n - \frac{(R_n - 1)R_n}{2} + \frac{R_n(R_n + 1)}{2} - n + 1 = R_n + 1.$$

We use this and the definitions of A and B to calculate

$$\begin{aligned} A\big(B(n)\big) &= A\Big(n - \frac{(R_n - 1)R_n}{2}, \frac{R_n(R_n + 1)}{2} - n + 1\Big) \\ &= \frac{\big(R_n + 1 - 1\big)\big(R_n + 1\big)}{2} + 1 - \Big(\frac{R_n(R_n + 1)}{2} - n + 1\Big) \\ &= n. \end{aligned}$$

This proves $A\big(B(n)\big) = n$ for all $n \in \mathbb{Z}^+$.

Thus (2) is proved, implying the $B : \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ is a bijection. This completes our rigorous proof that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.