In[114]:=
```
<<DiscreteMath`Combinatorica`
```

Problem 15

How many solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 21$?

$x_1 \geq 1$?

In[115]:=
```
Binomial[21 - 1 + 4, 4]
```

Out[115]=
```
10626
```

If $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 2$, $x_4 \geq 2$, $x_5 \geq 2$?

In[116]:=
```
Binomial[21 - 10 + 4, 4]
```

Out[116]=
```
1365
```

$x_1 \leq 10$?

In[117]:=
```
Binomial[21 + 4, 4] - Binomial[21 - 11 + 4, 4]
```

Out[117]=
```
11649
```

In[118]:=
```
Sum[Binomial[21 + 3 - k, 3], {k, 0, 10}]
```

Out[118]=
```
11649
```

$x_1 \leq 3$, $1 \leq x_2 \leq 3$ and $x_3 \geq 15$?

In[119]:=
```
x1g3 = Binomial[21 - 4 + 4, 4]
```

Out[119]=
```
5985
```

In[120]:=
```
x2r = Binomial[21 - 1 + 3, 3] + Binomial[21 - 2 + 3, 3] + Binomial[21 - 3 + 3, 3]
```

Out[120]=
```
4641
```

In[121]:=
```
x3g14 = Binomial[21 - 15 + 4, 4]
```
Out[121]=
```
210
```
In[122]:=
```
x3g14c = Binomial[(21 - 15 - 4) - 1 + 3, 3] +
  Binomial[(21 - 15 - 4) - 2 + 3, 3] + Binomial[(21 - 15 - 4) - 3 + 3, 3]
```
Out[122]=
```
5
```
In[123]:=
```
x3g14r = Binomial[(21 - 15) - 1 + 3, 3] +
  Binomial[(21 - 15) - 2 + 3, 3] + Binomial[(21 - 15) - 3 + 3, 3]
```
Out[123]=
```
111
```
In[124]:=
```
x3g14r - x3g14c
```
Out[124]=
```
106
```
Or, easier way:

restriction on x_3 and a part of x_2

In[125]:=
```
Binomial[(21 - 15 - 1) + 4, 4]
```
Out[125]=
```
126
```
How many solutions of $y_1 + y_2 + y_3 + y_4 + y_5 = 5$ such that $y_1 \leq 3$ and $y_2 \leq 2$? Look at the complement: How many solutions such that $y_1 > 3$ or $y_2 > 2$? These are disjoint sets.

In[126]:=
```
Binomial[(5 - 4) + 4, 4]
```
Out[126]=
```
5
```
In[127]:=
```
Binomial[(5 - 3) + 4, 4]
```
Out[127]=
```
15
```

Problem 16

How many solutions of $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$?

In[128]:=
```
Binomial[34, 5]
```
Out[128]=
```
278256
```

If $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 \geq 6$, $x_6 \geq 6$?

In[129]:=
```
Binomial[12, 5]
```
Out[129]=
```
792
```

$x_1 \leq 5$?

In[130]:=
```
Binomial[34, 5] - Binomial[28, 5]
```
Out[130]=
```
179976
```

In[131]:=
```
Sum[Binomial[33 - k, 4], {k, 0, 5}]
```
Out[131]=
```
179976
```

$x_1 \leq 7$ and $x_2 \geq 9$?

In[132]:=
```
x2g8 = Binomial[20 + 5, 5]
```
Out[132]=
```
53130
```

In[133]:=
```
x1g7 = Binomial[21 + 5, 5]
```
Out[133]=
```
65780
```

In[134]:=
```
x2g8x1g7 = Binomial[12 + 5, 5]
```
Out[134]=
```
6188
```

In[135]:=

```
x2g8 - x2g8x1g7
```

Out[135]=

```
46942
```

Problem 20

In[136]:=

```
Binomial[11 + 3, 3]
```

Out[136]=

```
364
```

In[137]:=

```
Length[Select[
  Flatten[Table[{x, y, z}, {x, 0, 11}, {y, 0, 11}, {z, 0, 11}], 2], (Apply[Plus, #] < 12) &]]
```

Out[137]=

```
364
```

Problem 23

In[138]:=

```
(* We are asked how many teams of two can be
  formed by 12 students. The teams have different colors. *)
```

In[139]:=

```
(* There are two ways to think about this. First,
 team the students following the permutations. But,
 this would count each team twice. *)
```

In[140]:=

$$\frac{12!}{(2!)^6}$$

Out[140]=

```
7484400
```

In[141]:=

```
(* The second way is by the product
  rule: There are 12 choose 2 ways of selecting the red team,
 there are 10 choose 2 ways of selecting green team,
 there are 8 choose 2 ways of selecting blue team, ... *)
```

In[142]:=

```
Product[Binomial[2 k, 2], {k, 6, 1, -1}]
```

Out[142]=

```
7484400
```

Problem 25

How many positive integers less than 10^6 have the sum of their digits equal to 19.

In[143]:=
```
(* this is a permutations with repetitions problem,
 but we are permited to put only digits 0,1,...,
 9 in six boxes. First we calculate all possibilities with nonnegative integers,
 then subtract those that contain 10 or larger integers. Fortunately
  10 or larger number can appear only at one spot *)
```

In[144]:=
```
Binomial[24, 5] - 6 Binomial[14, 5]
```

Out[144]=
```
30492
```

In[145]:=
```
Length[Select[IntegerDigits[#] & /@ Range[999999], (Apply[Plus, #] == 19) &]]
```

Out[145]=
```
30492
```

▪ How about the sum is 20

In[146]:=
```
Length[Select[IntegerDigits[#] & /@ Range[999999], (Apply[Plus, #] == 20) &]]
```

Out[146]=
```
35127
```

In[147]:=
```
Binomial[25, 5] - (6 Binomial[15, 5] - Binomial[6, 2])
```

Out[147]=
```
35127
```

▪ How about the sum is 21

In[148]:=
```
Length[Select[IntegerDigits[#] & /@ Range[999999], (Apply[Plus, #] == 21) &]]
```

Out[148]=
```
39662
```

In[149]:=
```
Binomial[26, 5] - (6 Binomial[16, 5] - 2 Binomial[6, 2])
```

Out[149]=
```
39602
```

Problem 26

How many positive integers less than 10^6 have the sum of their digits equal to 13 and exactly one digit equal to 9.

In[150]:=
```
(* total sum to 13 *)
```

In[151]:=
```
Binomial[13 + 5, 5] - 6 Binomial[3 + 5, 5]
```

Out[151]=
```
8232
```

In[152]:=
```
(* no 9 s *)
```

In[153]:=
```
Binomial[13 + 5, 5] - 6 Binomial[4 + 5, 5]
```

Out[153]=
```
7812
```

In[154]:=
```
(* the difference is one nine *)
```

In[155]:=
```
(Binomial[13 + 5, 5] - 6 Binomial[3 + 5, 5]) - (Binomial[13 + 5, 5] - 6 Binomial[4 + 5, 5])
```

Out[155]=
```
420
```

In[156]:=
```
(* different logic how meny with only 9 at first *)
```

In[157]:=
```
6 Binomial[4 + 4, 4]
```

Out[157]=
```
420
```

In[158]:=
```
Length[
 Select[IntegerDigits[#] & /@ Range[999999], (And[Apply[Plus, #] == 13, Max[#] == 9]) &]]
```

Out[158]=
```
420
```

Problem 27

There arc 10 questions on a discrete mathematics final exam. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

In[159]:=

```
Binomial[50 + 9, 9]
```

Out[159]=

```
12565671261
```

In[160]:=

```
Binomial[50 + 9, 9]
```

Out[160]=

```
12565671261
```

Problem 33

In[161]:=

```
3 (* words with one letter *) +
 (* words with two letters *) +
  1 (* two Os *) +
 2 * 2 (* one O *) +
 2 (* no Os *) +
 (* words with three letters *) +
  1 (* three Os *) +
 Binomial[3, 2] * 2 (* two Os *) +
 Binomial[3, 1] * 2 (* one O *) +
 (* words with four letters *) +
   Binomial[4, 3] * 2 (* three Os *) +
 Binomial[4, 2] * 2 (* two Os *) +
 (* words with five letters *)
 Binomial[5, 3] 2
```

Out[161]=

```
63
```

Notice

In[162]:=

```
5 ! / 3 !
```

Out[162]=

```
20
```

In[163]:=

```
Binomial[5, 3] Binomial[2, 1]
```

Out[163]=

```
20
```

A similar explanation.

In[164]:=
```
{O, R, O, N, O}
```
Out[164]=
```
{O, R, O, N, O}
```
In[165]:=
```
Binomial[3, 1] (* one letter *) + (3*2+1) (* two letters *) +
 (3*2*1+Binomial[3, 2]*2 +1) (* three letters *) +
 (Binomial[4, 2]*2 +Binomial[4, 3]*2) (* four letters *) + 5!/3!
```
Out[165]=
```
63
```

Problem 34

SEERESS

Strings of length 5

There are 3^5 = 243 strings of length 5 using these three characters. Out of these 243 strings there are 90 strings that use one of these 3 Es, 1 R and 1 S, or 3 Es, 0 R and 2 S, or 2 Es, 1 R and 2 Ss, or 2 Es, 0 R and 3 Ss, or 1 E, 1 R and 3 Ss. That is there are 90 strings of length 5 that can be formed from these 7 characters. Below is *Mathematica* proof of that. The corresponding calculation is as follows

In[166]:=

$$\frac{5!}{3!1!1!} + \frac{5!}{3!0!2!} + \frac{5!}{2!1!2!} + \frac{5!}{2!0!3!} + \frac{5!}{1!1!3!}$$

Out[166]=
```
90
```

Mathematica selects all such strings in the following command. We had to identify E to 0, R to 1, S to 2 to do the selection.

In[167]:=
```
Length[Select[Strings[{0, 1, 2}, 5], Function[y,
   Or[(Sort[y]) == {0, 0, 0, 1, 2}, (Sort[y]) == {0, 0, 0, 2, 2}, (Sort[y]) == {0, 0, 1, 2, 2},
    (Sort[y]) == {0, 0, 2, 2, 2}, (Sort[y]) == {0, 1, 2, 2, 2}]]]]
```
Out[167]=
```
90
```

Strings of length 6.

In[168]:=
```
3^6
```
Out[168]=
```
729
```

There are 3^5 = 729 strings of length 6 using these three characters. Out of these 729 strings there are 140 strings that use one of these 3 Es, 1 R and 2 Ss, or 3 Es, 0 R and 3 Ss, or 2 Es, 1 R and 3 Ss. That is there are 90 strings of length 5 that can be formed from these 7 characters. Below is *Mathematica* proof of that. The corresponding calculation is as follows

In[169]:=

$$\frac{6!}{3!\,1!\,2!} + \frac{6!}{3!\,0!\,3!} + \frac{6!}{2!\,1!\,3!}$$

Out[169]=

```
140
```

Mathematica selects all such strings in the following command. We had to identify E to 0, R to 1, S to 2 to do the selection.

In[170]:=

```
Length[Select[Strings[{0, 1, 2}, 6], Function[y, Or[(Sort[y]) == {0, 0, 0, 1, 2, 2},
    (Sort[y]) == {0, 0, 0, 2, 2, 2}, (Sort[y]) == {0, 0, 1, 2, 2, 2}]]]]
```

Out[170]=

```
140
```

Strings of length 7.

There are 140 such strings.

In[171]:=

$$\frac{7!}{3!\,1!\,3!}$$

Out[171]=

```
140
```

Mathematica selects all such strings in the following command. We had to identify E to 0, R to 1, S to 2 to do the selection.

In[172]:=

```
Length[Select[Strings[{0, 1, 2}, 7], Function[y, Or[(Sort[y]) == {0, 0, 0, 1, 2, 2, 2}]]]]
```

Out[172]=

```
140
```

Problem, similar smaller

FEEL

Strings of length 3

In[173]:=

```
3^3
```

Out[173]=

```
27
```

There are 3^3 = 27 strings of length 3 using these three characters. Out of these 27 strings there are ?? strings that use one of these 3 Es, 1 R and 1 S, or 3 Es, 0 R and 2 S, or 2 Es, 1 R and 2 Ss, or 2 Es, 0 R and 3 Ss, or 1 E, 1 R and 3 Ss. That is there are 90 strings of length 5 that can be formed from these 7 characters. Below is *Mathematica* proof of that. The corresponding calculation is as follows

In[174]:=

$$\frac{3!}{1!\,2!\,0!}+\frac{3!}{1!\,1!\,1!}+\frac{3!}{0!\,2!\,1!}$$

Out[174]=

12

Mathematica selects all such strings in the following command. We had to identify E to 0, F to 1, L to 2 in order to do the selection.

In[175]:=

```
Length[Select[Strings[{0, 1, 2}, 3],
  Function[y, Or[(Sort[y]) == {0, 1, 1}, (Sort[y]) == {0, 1, 2}, (Sort[y]) == {1, 1, 2}]]]]
```

Out[175]=

12

Strings of length 4.

In[176]:=

```
3^4
```

Out[176]=

81

There are 3^4 = 81 strings of length 4 using these three characters. Out of these 81 strings there are 12 strings that use 2 Es, 1 F and 1 L. Below is *Mathematica* proof of that. The corresponding calculation is as follows

In[177]:=

$$\frac{4!}{1!\,2!\,1!}$$

Out[177]=

12

Mathematica selects all such strings in the following command. We had to identify E to 0, R to 1, S to 2 to do the selection.

In[178]:=

```
Length[Select[Strings[{0, 1, 2}, 4], Function[y, Or[(Sort[y]) == {0, 1, 1, 2}]]]]
```

Out[178]=

12

Total 24.

Problem, similar smaller

SEEDS

Strings of length 3

There are 3^3 = 27 strings of length 3 using these three characters. Out of these 27 strings there are ?? strings that use one of these 3 Es, 1 R and 1 S, or 3 Es, 0 R and 2 S, or 2 Es, 1 R and 2 Ss, or 2 Es, 0 R and 3 Ss, or 1 E, 1 R and 3 Ss. That is there are 90 strings of length 5 that can be formed from these 7 characters. Below is *Mathematica* proof of that. The corresponding calculation is as follows

In[179]:=

$$\frac{3!}{1!\,2!\,0!}+\frac{3!}{1!\,1!\,1!}+\frac{3!}{0!\,2!\,1!}+\frac{3!}{0!\,1!\,2!}$$

Out[179]=

15

Mathematica selects all such strings in the following command. We had to identify D to 0, E to 1, S to 2 in order to do the selection.

In[180]:=

```
Length[Select[Strings[{0, 1, 2}, 3], Function[y, Or[(Sort[y]) == {0, 1, 1},
    (Sort[y]) == {0, 1, 2}, (Sort[y]) == {1, 1, 2}, (Sort[y]) == {1, 2, 2}]]]]
```

Out[180]=

15

Strings of length 4.

In[181]:=

```
3^4
```

Out[181]=

81

There are 3^4 = 81 strings of length 4 using these three characters. ??? strings of length 4 that can be formed from these 7 characters. Below is *Mathematica* proof of that. The corresponding calculation is as follows

In[182]:=

$$\frac{4!}{1!\,2!\,1!}+\frac{4!}{1!\,1!\,2!}+\frac{4!}{0!\,2!\,2!}$$

Out[182]=

30

Mathematica selects all such strings in the following command. We had to identify E to 0, R to 1, S to 2 to do the selection.

In[183]:=
```
Length[Select[Strings[{0, 1, 2}, 4], Function[y,
   Or[(Sort[y]) == {0, 1, 1, 2}, (Sort[y]) == {0, 1, 2, 2}, (Sort[y]) == {1, 1, 2, 2}]]]]
```

Out[183]=
```
30
```

Strings of length 5.

There are 140 such strings.

In[184]:=

$$\frac{5!}{1!\,2!\,2!}$$

Out[184]=
```
30
```

Mathematica selects all such strings in the following command. We had to identify E to 0, R to 1, S to 2 to do the selection.

In[185]:=
```
Length[Select[Strings[{0, 1, 2}, 5], Function[y, Or[(Sort[y]) == {0, 1, 1, 2, 2}]]]]
```

Out[185]=
```
30
```

Total 75.

Problem 35

EVERGREEN

(4)E to 0, (1) G to 1, (1)N to 2, (2)R to 3, (1)V to 4

Strings of length 7

In[186]:=
```
5^7
```

Out[186]=
```
78125
```

There are 5^7 = 78125 strings of length 7 using these five characters. Out of these 78125 strings there are ?? strings that use appropriate counts of each letter. That is there are 90 strings of length 5 that can be formed from these 7 characters. Below is *Mathematica* proof of that. The corresponding calculation is as follows

In[187]:=

$$\frac{7!}{4!1!1!1!0!}+\frac{7!}{4!1!1!0!1!}+\frac{7!}{4!1!0!2!0!}+\frac{7!}{4!1!0!1!1!}+\frac{7!}{4!0!1!1!1!}+\frac{7!}{4!0!0!2!1!}+\frac{7!}{3!1!1!2!0!}+\frac{7!}{3!1!1!1!1!}+\frac{7!}{3!1!0!2!1!}+\frac{7!}{3!0!1!2!1!}+\frac{7!}{2!1!1!2!1!}$$

Out[187]=

4410

Mathematica selects all such strings in the following command. We had to identify E to 0, G to 1, N to 2, R to 3, V to 4 in oeder to do the selection.

In[188]:=

```
Length[Select[Strings[{0, 1, 2, 3, 4}, 7],
  Function[y, Or[(Sort[y]) == {0, 0, 0, 0, 1, 2, 3}, (Sort[y]) == {0, 0, 0, 0, 1, 2, 4},
    (Sort[y]) == {0, 0, 0, 0, 1, 3, 3}, (Sort[y]) == {0, 0, 0, 0, 1, 3, 4},
    (Sort[y]) == {0, 0, 0, 0, 2, 3, 4}, (Sort[y]) == {0, 0, 0, 0, 3, 3, 4},
    (Sort[y]) == {0, 0, 0, 1, 2, 3, 3}, (Sort[y]) == {0, 0, 0, 1, 2, 3, 4},
    (Sort[y]) == {0, 0, 0, 1, 3, 3, 4}, (Sort[y]) == {0, 0, 0, 2, 3, 3, 4},
    (Sort[y]) == {0, 0, 1, 2, 3, 3, 4}]]]]
```

4410

Strings of length 8.

```
5^8
```

390625

There are 5^8 = 390625 strings of length 8 using these five characters. Out of these 390625 strings there are ???? strings that use appropriate number of each letter.   Below is *Mathematica* proof of that. The corresponding calculation is as follows

$$\frac{8!}{4!1!1!2!0!}+\frac{8!}{4!1!1!1!1!}+\frac{8!}{4!1!0!2!1!}+\frac{8!}{4!0!1!2!1!}+\frac{8!}{3!1!1!2!1!}$$

7560

Mathematica selects all such strings in the following command. We had to identify E to 0, G to 1, N to 2, R to 3, V to 4 in oeder to do the selection.

```
Length[Select[Strings[{0, 1, 2, 3, 4}, 8],
  Function[y, Or[(Sort[y]) == {0, 0, 0, 0, 1, 2, 3, 3},
    (Sort[y]) == {0, 0, 0, 0, 1, 2, 3, 4}, (Sort[y]) == {0, 0, 0, 0, 1, 3, 3, 4},
    (Sort[y]) == {0, 0, 0, 0, 2, 3, 3, 4}, (Sort[y]) == {0, 0, 0, 1, 2, 3, 3, 4}]]]]
```

7560

Strings of length 9.

```
5^9
```

```
1953125
```

There are 5^9 = 1953125 such strings

There are 7560 strings which use 4 Es, 1 G, 1 N, 2 Rs, 1 V.

$$\frac{9!}{4!\,1!\,1!\,2!\,1!}$$

```
7560
```

Mathematica selects all such strings in the following command. We had to identify E to 0, G to 1, N to 2, R to 3, V to 4 in oeder to do the selection.

```
Length[Select[Strings[{0, 1, 2, 3, 4}, 9],
  Function[y, Or[(Sort[y]) == {0, 0, 0, 0, 1, 2, 3, 3, 4}]]]]
```

```
7560
```

Problem 39

We have to make 12 steps: 4 in x drection, 3 in y direction and 5 in z direction. This is exactly the same as counting the number of strings of 12 characters {x, x, x, x, y, y, y, z, z, z, z, z}

$$\frac{(4+3+5)!}{3!\,4!\,5!}$$

```
27720
```

For a 3x3x3 cube that would be:

$$\frac{(3+3+3)!}{3!\,3!\,3!}$$

```
1680
```