In[ ]:= `NotebookDirectory[]`

Equilibrium temperature distribution - 2D problem

The problem

The objective is to solve the PDE

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \text{ on } \{(x, y) \in \mid 0 \le x \le K,\ 0 \le y \le L\},$$

subject to the conditions

$u(x, 0) = f_1(x),\ u(x, L) = f_2(x)$ (call these **Boundary Conditions for x**)

$u(0, y) = g_1(y),\ u(K, y) = g_2(y)$ (call these **Boundary Conditions for y**)

The trick is to split this problem into **two problems**

Problem 1

In this part, Problem 1, the objective is to solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } 0 \le x \le K,\ 0 \le y \le L,$$

subject to the conditions

$u(x, 0) = f_1(x), \; u(x, L) = f_2(x)$ (call these **Boundary Conditions for x**)

$u(0, y) = 0, \; u(K, y) = 0$ (call these **Zero Boundary Conditions for y**)

Step 1. First ignore **Boundary Conditions for x** and solve

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on $0 \le x \le K, \; 0 \le y \le L,$

subject to the conditions

$u(0, y) = 0, \; u(K, y) = 0$ (**Zero Boundary Conditions for y**)

Using the Separation of Variables (SofV) method we **find a "few" solutions** of this problem. These are two sequences of solutions: (below $n \in \mathbb{N}$)

$$\mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}\,x\right]\frac{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\,y\right]}{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\,L\right]}$$

$$\mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}\,x\right]\frac{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\,(L-y)\right]}{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\,L\right]}$$

Test these solutions: (the command below **(D[#,{x,2}]+D[#,{y,2}])&[ ]** calculates the second partial derivatives and adds them, the function is inclosed in the square brackets)

In[13]:= $(\mathrm{D}[\#, \{x, 2\}] + \mathrm{D}[\#, \{y, 2\}])\,\&\left[\mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}\,x\right]\frac{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\,y\right]}{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\,L\right]}\right]$

Out[13]= 0

Check the boundary conditions:

```
In[1]:= FullSimplify[(Sin[n Pi/K x] Sinh[n Pi/K y]/Sinh[n Pi/K L]) /. {x → {0, K}}]
```

```
Out[1]= {0, Csch[L n π/K] Sin[n π] Sinh[n π y/K]}
```

We need to tell *Mathematica* that n is an integer.

```
In[2]:= FullSimplify[(Sin[n Pi/K x] Sinh[n Pi/K y]/Sinh[n Pi/K L]) /. {x → {0, K}},
  n ∈ Integers]
```

```
Out[2]= {0, 0}
```

Verify the sequence of second solutions:

```
In[7]:= (D[#, {x, 2}] + D[#, {y, 2}]) &[Sin[n Pi/K x] Sinh[n Pi/K (L - y)]/Sinh[n Pi/K L]]
```

```
Out[7]= 0
```

```
In[8]:= FullSimplify[(Sin[n Pi/K x] Sinh[n Pi/K (L - y)]/Sinh[n Pi/K L]) /. {x → {0, K}},
  n ∈ Integers]
```

```
Out[8]= {0, 0}
```

Step 2. Now that we have a “few” solutions, we form many solutions. This is commonly known as the superposition principle:

$$u_1(x, y) = \sum_{n=1}^{nn} a_n \,\mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}\, x\right] \frac{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\, y\right]}{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\, L\right]} +$$

$$\sum_{n=1}^{nn} b_n \,\mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}\, x\right] \frac{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\,(L - y)\right]}{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\, L\right]}$$

Next we choose a_n and b_n such that the above function satisfies **Boundary Conditions for x**. First substitute $y = 0$. This leads to the formula for b_n.

$$f_1(x) = u(x, 0) = \sum_{n=1}^{nn} a_n \,\mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}\, x\right] \frac{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\, 0\right]}{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\, L\right]} +$$

$$\sum_{n=1}^{nn} b_n \,\mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}\, x\right] \frac{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\,(L - 0)\right]}{\mathrm{Sinh}\left[\frac{n\,\mathrm{Pi}}{K}\, L\right]} = \sum_{n=1}^{nn} b_n \,\mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}\, x\right]$$

Thus

$$f_1(x) = \sum_{n=1}^{nn} b_n \,\mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}\, x\right]$$

Multiply both sides with $\mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}\, x\right]$

$$f_1(x)\,\mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}\, x\right] = \sum_{n=1}^{nn} b_n \,\mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}\, x\right] \mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}\, x\right]$$

Integrate both sides from 0 to K. We need the following integrals:

$$\int_0^K f_1(x)\,\mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}\, x\right] dx =$$

$$\sum_{n=1}^{nn} b_n \int_0^K \mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}\, x\right] \mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}\, x\right] dx$$

In[9]:=
```
Clear[K];
FullSimplify[Integrate[Sin[n Pi/K x] Sin[j Pi/K x], {x, 0, K}],
 And[n ∈ Integers, j ∈ Integers, Or[j > n, j < n]]]
```

Out[9]= 0

In[10]:=
```
Clear[K];
FullSimplify[Integrate[Sin[n Pi/K x] Sin[n Pi/K x], {x, 0, K}],
 And[n ∈ Integers]]
```

Out[10]= $\frac{K}{2}$

Based on the above two integral calculations we have

$$\int_0^K f_1(x)\,\mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}x\right]dx =$$

$$\sum_{n=1}^{nn} b_n \int_0^K \mathrm{Sin}\left[\frac{n\,\mathrm{Pi}}{K}x\right]\mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}x\right]dx =$$

$$b_j \int_0^K \mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}x\right]\mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}x\right]dx$$

$$\int_0^K f_1(x)\,\mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}x\right]dx = b_j \int_0^K \mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}x\right]\mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}x\right]dx$$

$$\int_0^K f_1(x)\,\mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}x\right]dx = b_j\,\frac{K}{2}$$

Hence

$$b_j = \frac{2}{K}\int_0^K f_1(x)\,\mathrm{Sin}\left[\frac{j\,\mathrm{Pi}}{K}x\right]dx \quad (\text{we have the formula for } b_j)$$

Step 3. Now we substitute y = L in the formula for the solution. This will give us a_n

$$f_2(x) = u(x, L) =$$

$$\sum_{n=1}^{nn} a_n \operatorname{Sin}\left[\frac{n\,Pi}{K} x\right] \frac{\operatorname{Sinh}\left[\frac{n\,Pi}{K} L\right]}{\operatorname{Sinh}\left[\frac{n\,Pi}{K} L\right]} + \sum_{n=1}^{nn} b_n \operatorname{Sin}\left[\frac{n\,Pi}{K} x\right]$$

$$\frac{\operatorname{Sinh}\left[\frac{n\,Pi}{K} (L - L)\right]}{\operatorname{Sinh}\left[\frac{n\,Pi}{K} L\right]} = \sum_{n=1}^{nn} a_n \operatorname{Sin}\left[\frac{n\,Pi}{K} x\right]$$

Thus

$$f_2(x) = \sum_{n=1}^{nn} a_n \operatorname{Sin}\left[\frac{n\,Pi}{K} x\right]$$

Multiply both sides with $\operatorname{Sin}\left[\frac{j\,Pi}{K} x\right]$

$$f_2(x) \operatorname{Sin}\left[\frac{j\,Pi}{K} x\right] = \sum_{n=1}^{nn} a_n \operatorname{Sin}\left[\frac{n\,Pi}{K} x\right] \operatorname{Sin}\left[\frac{j\,Pi}{K} x\right]$$

Integrate both sides from 0 to K. We need the following integrals:

$$\int_0^K f_2(x) \operatorname{Sin}\left[\frac{j\,Pi}{K} x\right] dx =$$

$$\sum_{n=1}^{nn} a_n \int_0^K \operatorname{Sin}\left[\frac{n\,Pi}{K} x\right] \operatorname{Sin}\left[\frac{j\,Pi}{K} x\right] dx$$

We need the following integral in the sum:

In[11]:=
```
Clear[K];
FullSimplify[Integrate[Sin[n Pi/K x] Sin[j Pi/K x], {x, 0, K}],
 And[n ∈ Integers, j ∈ Integers, Or[j > n, j < n]]]
```

Out[11]= 0

In[12]:=
```
Clear[K];
FullSimplify[Integrate[Sin[n Pi/K x] Sin[n Pi/K x], {x, 0, K}],
 And[n ∈ Integers]]
```

Out[12]= $\frac{K}{2}$

Based on the above two integral calculations we have

$$\int_0^K f_2(x)\,\text{Sin}\left[\frac{j\,\text{Pi}}{K}x\right]dx =$$

$$\sum_{n=1}^{nn} a_n \int_0^K \text{Sin}\left[\frac{n\,\text{Pi}}{K}x\right]\text{Sin}\left[\frac{j\,\text{Pi}}{K}x\right]dx =$$

$$a_j \int_0^K \text{Sin}\left[\frac{j\,\text{Pi}}{K}x\right]\text{Sin}\left[\frac{j\,\text{Pi}}{K}x\right]dx$$

$$\int_0^K f_2(x)\,\text{Sin}\left[\frac{j\,\text{Pi}}{K}x\right]dx = a_j \int_0^K \text{Sin}\left[\frac{j\,\text{Pi}}{K}x\right]\text{Sin}\left[\frac{j\,\text{Pi}}{K}x\right]dx$$

$$\int_0^K f_2(x)\,\text{Sin}\left[\frac{j\,\text{Pi}}{K}x\right]dx = a_j\,\frac{K}{2}$$

Hence

$$a_j = \frac{2}{K}\int_0^K f_2(x)\,\text{Sin}\left[\frac{j\,\text{Pi}}{K}x\right]dx \quad (\text{we have the formula for } a_j)$$

Problem 2

In this part, Problem 2, the objective is to solve the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } 0 \le x \le K,\ 0 \le y \le L,$$

subject to the conditions

$u(x, 0) = 0,\ u(x, L) = 0$ (**Zero Boundary Conditions for x**)

$u(0, y) = g_1(y),\ u(K, y) = g_2(y)$ (**Boundary Conditions for y**)

Step 1. First ignore the **Boundary Conditions for y** and solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } 0 \le x \le K,\ 0 \le y \le L,$$

subject to the conditions

$u(x, 0) = 0,\ u(x, L) = 0$ (**Zero Boundary Conditions for x**)

Using Separation of Variables method we **find a "few" solutions of this problem**:

$$\text{Sin}\left[\frac{n\,\text{Pi}}{L}\,y\right]\frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{L}\,x\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{L}\,K\right]}$$

$$\text{Sin}\left[\frac{n\,\text{Pi}}{L}\,y\right]\frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{L}\,(K - x)\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{L}\,K\right]}$$

Test these solutions:

The first sequence

In[14]:= $(\text{D}[\#, \{x, 2\}] + \text{D}[\#, \{y, 2\}])\ \&\left[\text{Sin}\left[\frac{n\,\text{Pi}}{L}\,y\right]\frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{L}\,x\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{L}\,K\right]}\right]$

Out[14]= 0

```
In[15]:= FullSimplify[(Sin[n Pi/L y] Sinh[n Pi/L x]/Sinh[n Pi/L K]) /. {y → {0, L}},
  n ∈ Integers]

Out[15]= {0, 0}
```

The second sequence

```
In[16]:= (D[#, {x, 2}] + D[#, {y, 2}]) &[Sin[n Pi/L y] Sinh[n Pi/L (K - x)]/Sinh[n Pi/L K]]

Out[16]= 0

In[17]:= FullSimplify[(Sin[n Pi/L y] Sinh[n Pi/L (K - x)]/Sinh[n Pi/L K]) /. {y → {0, L}},
  n ∈ Integers]

Out[17]= {0, 0}
```

Step 2. Now that we have a “few” solutions we form many solutions. This is commonly known as the **superposition principle**:

$$u_2(x, y) = \sum_{n=1}^{nn} c_n \operatorname{Sin}\left[\frac{n\,Pi}{L} y\right] \frac{\operatorname{Sinh}\left[\frac{n\,Pi}{L} x\right]}{\operatorname{Sinh}\left[\frac{n\,Pi}{L} K\right]} + \sum_{n=1}^{nn} d_n \operatorname{Sin}\left[\frac{n\,Pi}{L} y\right] \frac{\operatorname{Sinh}\left[\frac{n\,Pi}{L} (K - x)\right]}{\operatorname{Sinh}\left[\frac{n\,Pi}{L} K\right]}$$

Now we choose c_n and d_n such that the above function satisfies the **Boundary Conditions for** conditions. First substitute $x = 0$. This leads to the formula for d_n. Then substitute $x = K$. This leads to the formula for c_n.

Similarly as in Problem 1, we first set $x = 0$:

$$g_1(y) = u_2(0, u) = \sum_{n=1}^{nn} c_n \, \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right] \frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{L} 0\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{L} K\right]} + \sum_{n=1}^{nn} d_n \, \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right] \frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{L} (K - 0)\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{L} K\right]} = \sum_{n=1}^{nn} d_n \, \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right]$$

Hence,

$$g_1(y) = \sum_{n=1}^{nn} d_n \, \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right]$$

Multiply both sides with $\text{Sin}\left[\frac{j\,\text{Pi}}{L} y\right]$

$$g_1(y) \, \text{Sin}\left[\frac{j\,\text{Pi}}{L} y\right] = \sum_{n=1}^{nn} d_n \, \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right] \text{Sin}\left[\frac{j\,\text{Pi}}{L} y\right]$$

Integrate both sides from 0 to L. We need the following integrals:

$$\int_0^L g_1(y) \, \text{Sin}\left[\frac{j\,\text{Pi}}{L} y\right] dy = \sum_{n=1}^{nn} d_n \int_0^L \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right] \text{Sin}\left[\frac{j\,\text{Pi}}{L} y\right] dy$$

We need the following integral in the sum:

In[18]:=
```
Clear[L];
FullSimplify[Integrate[Sin[n Pi/L y] Sin[j Pi/L y], {y, 0, L}],
 And[n ∈ Integers, j ∈ Integers, Or[j > n, j < n]]]
```

Out[18]= 0

In[19]:=
```
Clear[L];
FullSimplify[Integrate[Sin[n Pi/L y] Sin[n Pi/L y], {y, 0, L}],
 And[n ∈ Integers]]
```

Out[19]= $\frac{L}{2}$

Based on the above two integral calculations we have

$$\int_0^L g_1(y)\,\text{Sin}\left[\frac{j\,\text{Pi}}{L}y\right]dy =$$

$$\sum_{n=1}^{nn} d_n \int_0^L \text{Sin}\left[\frac{n\,\text{Pi}}{L}y\right]\text{Sin}\left[\frac{j\,\text{Pi}}{L}y\right]dy =$$

$$d_j \int_0^L \text{Sin}\left[\frac{j\,\text{Pi}}{L}y\right]\text{Sin}\left[\frac{j\,\text{Pi}}{L}y\right]dy$$

Hence:

$$\int_0^L g_1(y)\,\text{Sin}\left[\frac{j\,\text{Pi}}{L}y\right]dy = d_j \int_0^L \text{Sin}\left[\frac{j\,\text{Pi}}{L}y\right]\text{Sin}\left[\frac{j\,\text{Pi}}{L}y\right]dy$$

$$\int_0^L g_1(y)\,\text{Sin}\left[\frac{j\,\text{Pi}}{L}y\right]dy = d_j\,\frac{L}{2}$$

Hence

$$d_j = \frac{2}{L}\int_0^L g_1(y)\,\text{Sin}\left[\frac{j\,\text{Pi}}{L}y\right]dy \quad \left(\text{we have the formula for } d_j\right)$$

Similarly we calculate c_j: **we set** $x = K$:

$$g_2(y) = u_2(K, u) = \sum_{n=1}^{nn} c_n \, \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right] \frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{L} K\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{L} K\right]} +$$

$$\sum_{n=1}^{nn} d_n \, \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right] \frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{L} (K - K)\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{L} K\right]} = \sum_{n=1}^{nn} c_n \, \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right]$$

Hence,

$$g_2(y) = \sum_{n=1}^{nn} c_n \, \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right]$$

Multiply both sides with $\text{Sin}\left[\frac{j\,\text{Pi}}{L} y\right]$

$$g_2(y) \, \text{Sin}\left[\frac{j\,\text{Pi}}{L} y\right] = \sum_{n=1}^{nn} c_n \, \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right] \text{Sin}\left[\frac{j\,\text{Pi}}{L} y\right]$$

Integrate both sides from 0 to L. We need the following integrals:

$$\int_0^L g_2(y) \, \text{Sin}\left[\frac{j\,\text{Pi}}{L} y\right] dy =$$

$$\sum_{n=1}^{nn} c_n \int_0^L \text{Sin}\left[\frac{n\,\text{Pi}}{L} y\right] \text{Sin}\left[\frac{j\,\text{Pi}}{L} y\right] dy$$

We need the following integral in the sum:

In[20]:=
```
Clear[L];
FullSimplify[Integrate[Sin[n Pi/L y] Sin[j Pi/L y], {y, 0, L}],
 And[n ∈ Integers, j ∈ Integers, Or[j > n, j < n]]]
```

Out[20]= 0

In[21]:=
```
Clear[L];
FullSimplify[Integrate[Sin[n Pi/L y] Sin[n Pi/L y], {y, 0, L}],
 And[n ∈ Integers]]
```

Out[21]= $\frac{L}{2}$

Based on the above two integral calculations we have

$$\int_0^L g_2(y)\,\text{Sin}\left[\frac{j\,\text{Pi}}{L}\,y\right]\,dy =$$

$$\sum_{n=1}^{nn} c_n \int_0^L \text{Sin}\left[\frac{n\,\text{Pi}}{L}\,y\right]\,\text{Sin}\left[\frac{j\,\text{Pi}}{L}\,y\right]\,dy =$$

$$c_j \int_0^L \text{Sin}\left[\frac{j\,\text{Pi}}{L}\,y\right]\,\text{Sin}\left[\frac{j\,\text{Pi}}{L}\,y\right]\,dy$$

Hence:

$$\int_0^L g_2(y)\,\text{Sin}\left[\frac{j\,\text{Pi}}{L}\,y\right]\,dy = c_j \int_0^L \text{Sin}\left[\frac{j\,\text{Pi}}{L}\,y\right]\,\text{Sin}\left[\frac{j\,\text{Pi}}{L}\,y\right]\,dy$$

$$\int_0^L g_2(y)\,\text{Sin}\left[\frac{j\,\text{Pi}}{L}\,y\right]\,dy = c_j\,\frac{L}{2}$$

Hence

$$c_j = \frac{2}{L}\int_0^L g_2(y)\,\text{Sin}\left[\frac{j\,\text{Pi}}{L}\,y\right]\,dy \quad (\text{we have the formula for } c_j)$$

A symbolic implementation

Here are the given quantities

In[22]:=
```
Clear[lK1, lL1, f11, f21, g11, g21, nn1];

nn1 = 15;

lK1 = 1; lL1 = 1;

f11[x_] = 4 x^2 (1 - x);

g21[y_] = 4 y (1 - y)^2;

f21[x_] = 0;

g11[y_] = 0;
```

In[29]:=
```
Clear[aa1];

aa1[n_] =
 FullSimplify[
  2/lK1 Integrate[f21[x] Sin[n Pi/lK1 x], {x, 0, lK1}],
  And[n ∈ Integers, n > 0]]
```

Out[30]= 0

```
In[31]:= Clear[bb1];

bb1[n_] =
 FullSimplify[
  2/lK1 Integrate[f11[x] Sin[n Pi/lK1 x], {x, 0, lK1}],
  And[n ∈ Integers, n > 0]]
```

Out[32]= $-\frac{16\,(1+2\,(-1)^n)}{n^3\,\pi^3}$

```
In[33]:= Clear[cc1];

cc1[n_] =
 FullSimplify[
  2/lL1 Integrate[g21[y] Sin[n Pi/lL1 y], {y, 0, lL1}],
  And[n ∈ Integers, n > 0]]
```

Out[34]= $\frac{16\,(2+(-1)^n)}{n^3\,\pi^3}$

```
In[35]:= Clear[dd1];

dd1[n_] =
 FullSimplify[
  2/lL1 Integrate[g11[y] Sin[n Pi/lL1 y], {y, 0, lL1}],
  And[n ∈ Integers, n > 0]]
```

Out[36]= 0

The solution is

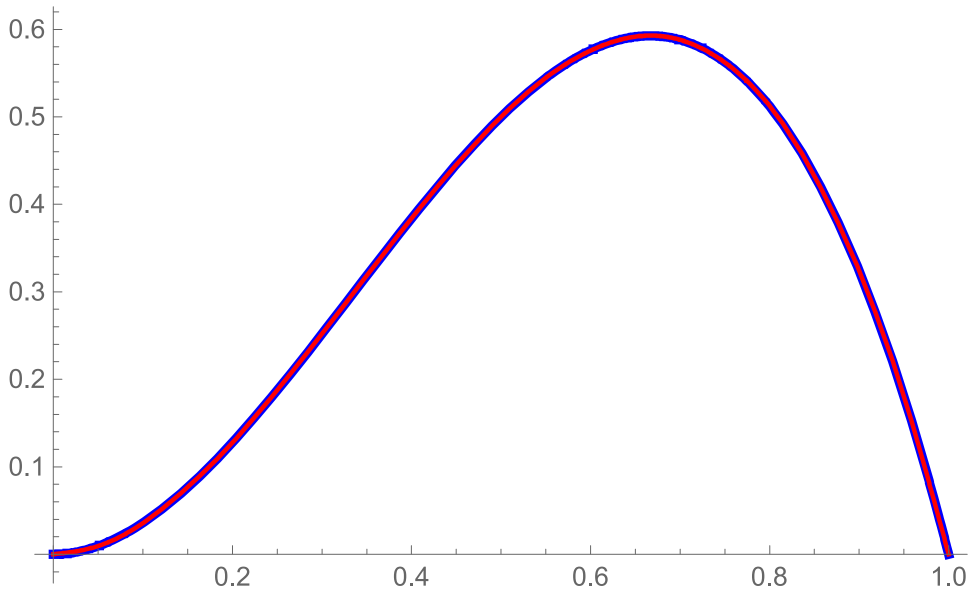
```
In[37]:= Clear[uu1];
```

$$uu1[x\_, y\_] = \sum_{n=1}^{nn1} aa1[n] \, Sin\left[\frac{n\,Pi}{lK1}\,x\right] \frac{Sinh\left[\frac{n\,Pi}{lK1}\,y\right]}{Sinh\left[\frac{n\,Pi}{lK1}\,lL1\right]} + \sum_{n=1}^{nn1} bb1[n] \, Sin\left[\frac{n\,Pi}{lK1}\,x\right] \frac{Sinh\left[\frac{n\,Pi}{lK1}\,(lL1 - y)\right]}{Sinh\left[\frac{n\,Pi}{lK1}\,lL1\right]} + \sum_{n=1}^{nn1} cc1[n] \, Sin\left[\frac{n\,Pi}{lL1}\,y\right] \frac{Sinh\left[\frac{n\,Pi}{lL1}\,x\right]}{Sinh\left[\frac{n\,Pi}{lL1}\,lK1\right]} + \sum_{n=1}^{nn1} dd1[n] \, Sin\left[\frac{n\,Pi}{lL1}\,y\right] \frac{Sinh\left[\frac{n\,Pi}{lL1}\,(lK1 - x)\right]}{Sinh\left[\frac{n\,Pi}{lL1}\,lK1\right]};$$

How good is our approximation for the function f1[x] in the boundary conditions? Here is a visual answer.

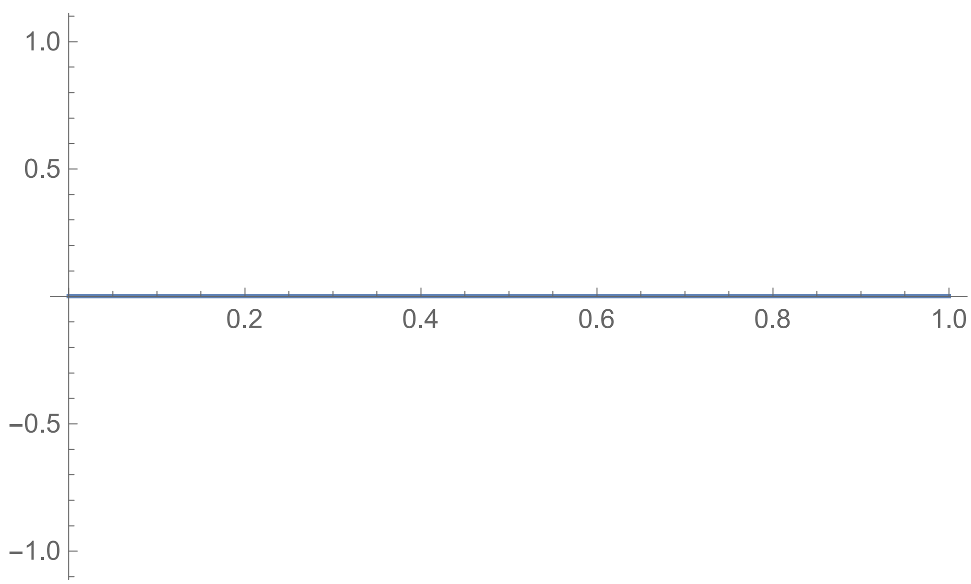
In[45]:=
```
Plot[{f11[x], uu1[x, 0]}, {x, 0, lK1},
 PlotStyle → {{Blue, Thickness[0.01]},
   {Red, Thickness[0.005]}}]
```

Out[45]=



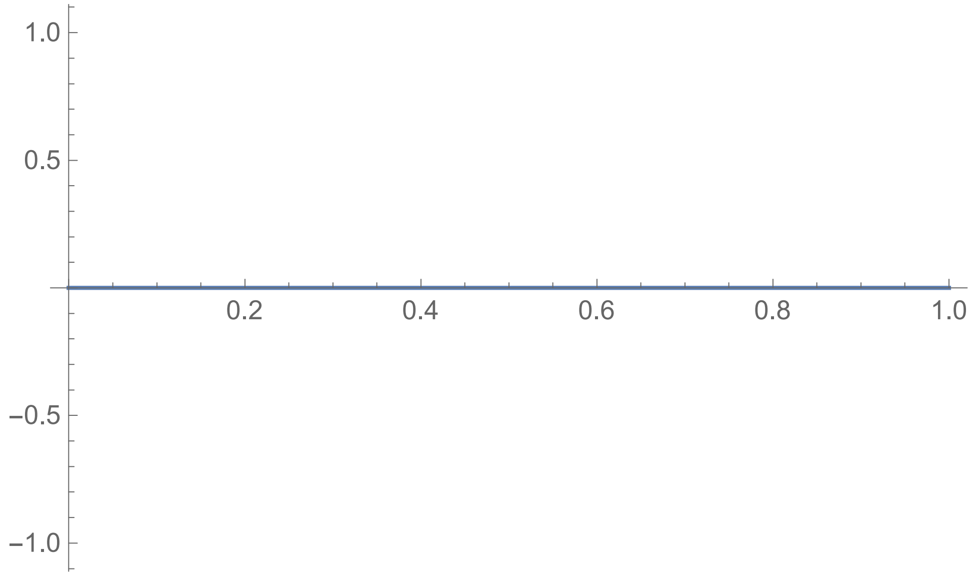
In[46]:=
```
Plot[uu1[x, lL1], {x, 0, lK1}]
```

Out[46]=



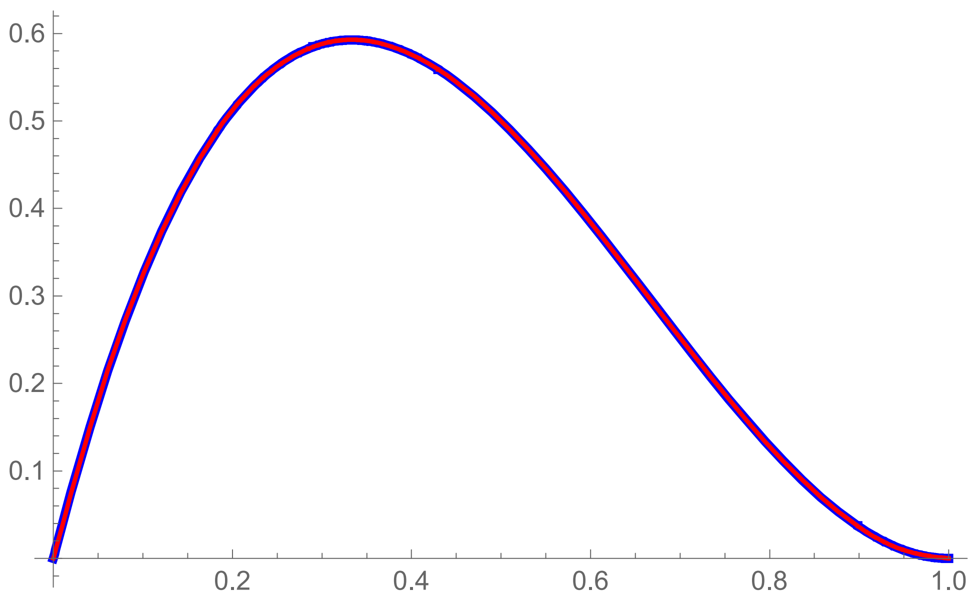
In[47]:=
```
Plot[uu1[0, y], {y, 0, lL1}]
```

Out[47]=



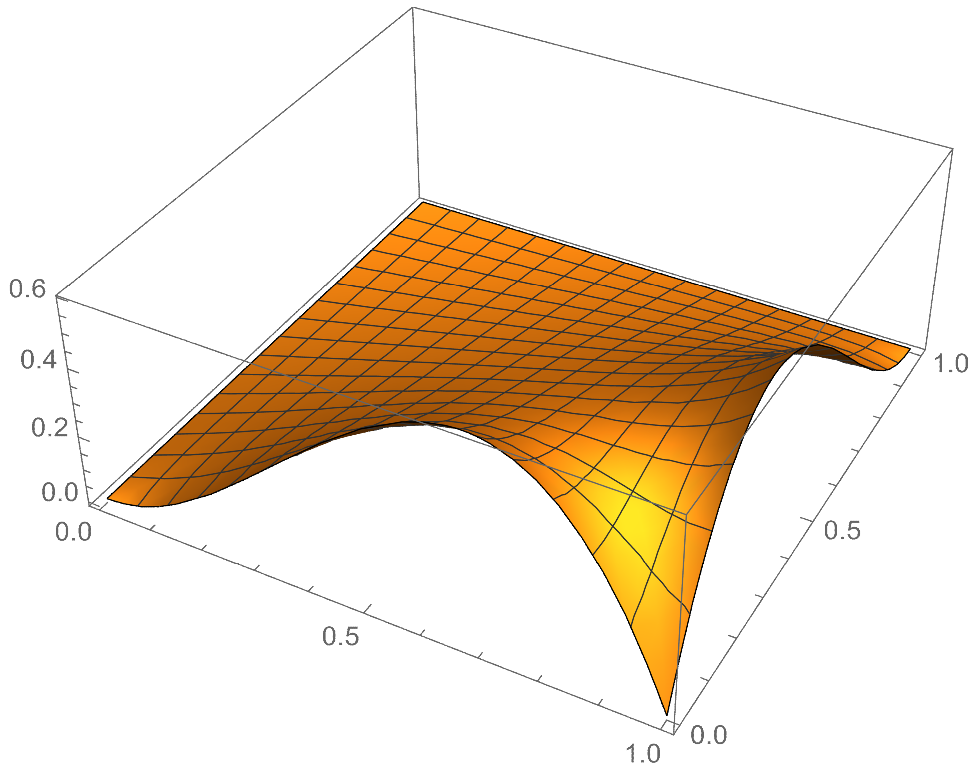
In[48]:=
```
Plot[{g21[y], uu1[lK1, y]}, {y, 0, lL1},
 PlotStyle → {{Blue, Thickness[0.01]},
   {Red, Thickness[0.005]}}]
```

Out[48]=



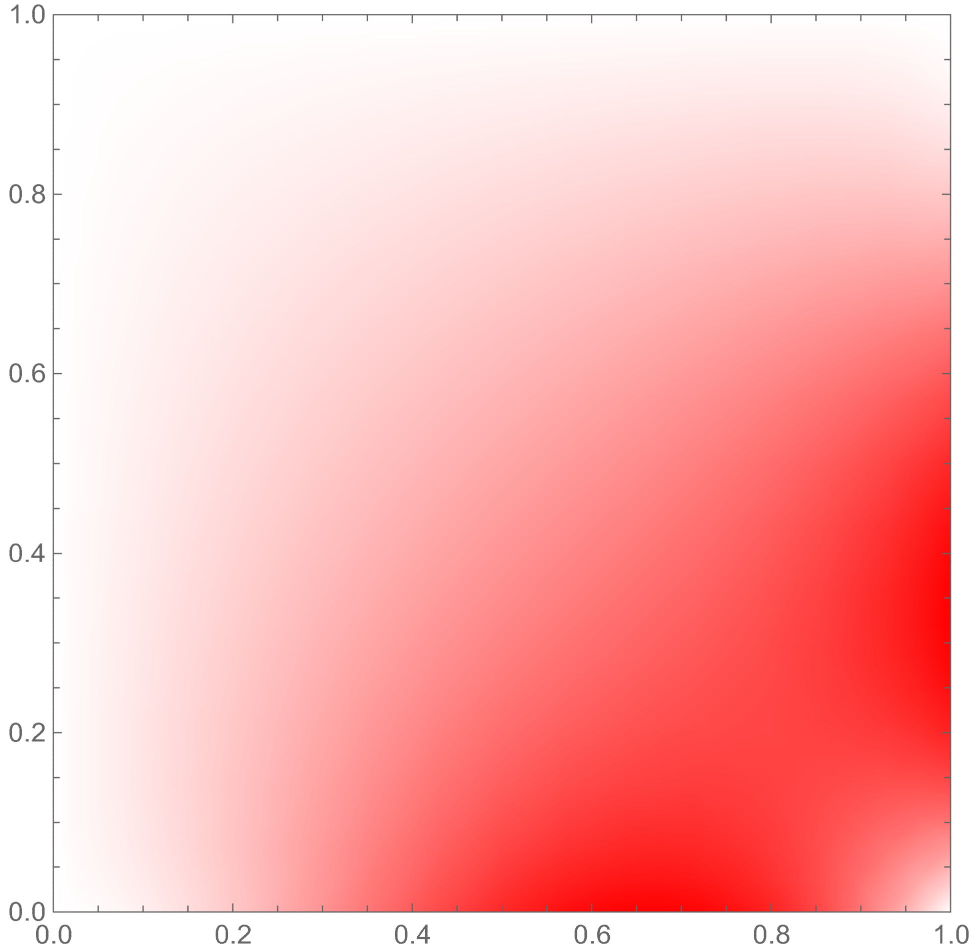
In[49]:=
```
Plot3D[N[uu1[x, y]], {x, 0, lK1}, {y, 0, lL1},
 Mesh → Automatic]
```

Out[49]=



In[50]:=
```
DensityPlot[N[uu1[x, y]], {x, 0, lK1}, {y, 0, lL1},
 Frame → True, PlotRange → {{0, 1}, {0, 1}},
 ColorFunction → (RGBColor[1, 1 - #, 1 - #] &)]
```

Out[50]=



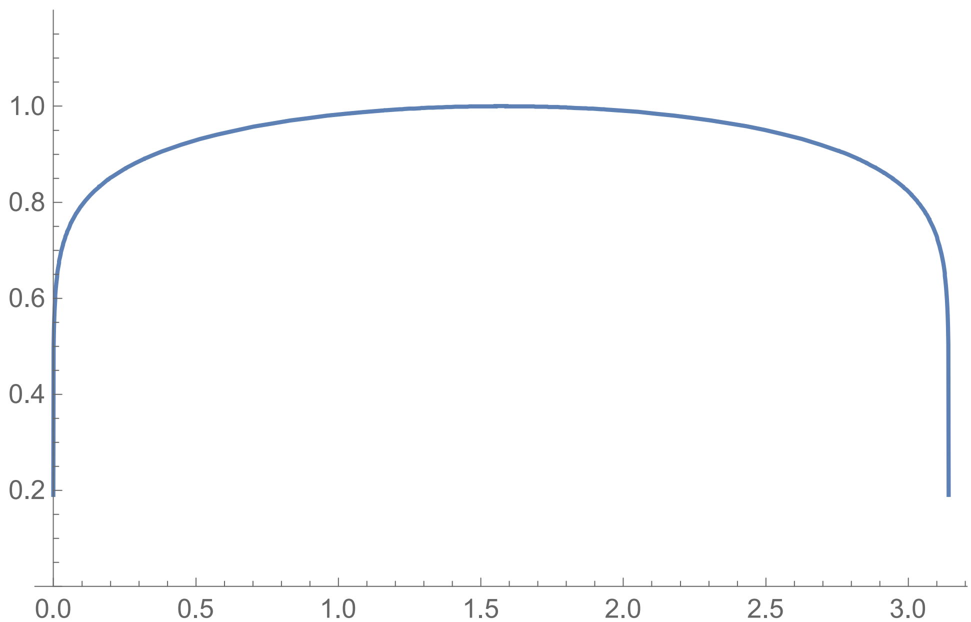
A numerical implementation

Here are the given quantities

In[51]:=
```
Clear[lK2, lL2, f12, f22, g12, g22, nn2];

nn2 = 45;

lK2 = Pi; lL2 = Pi;

f12[x_] = (Sin[x])^(1/10);
g22[y_] = 0; f22[x_] = 0;
g12[y_] = 0;
```

In[55]:=
```
Plot[f12[x], {x, 0, lK2}, PlotRange → {0, 1.2}]
```

Out[55]=



In[56]:=
```
Clear[aa2];

aa2[n_] = 2/lK2 Integrate[f22[x] Sin[n Pi/lK2 x], {x, 0, lK2}]
```

Out[57]= 0

In[58]:=
```
Clear[bbl2];

bbl2 =
  Chop[Table[2/lK2 NIntegrate[f12[x] Sin[n Pi/lK2 x], {x, 0, lK2},
      Method → {Automatic}, MaxRecursion → 200,
      AccuracyGoal → 12, PrecisionGoal → 16], {n, 1, nn2}]]
```

Out[59]=
```
{1.23582, 0, 0.358787, 0, 0.204016, 0, 0.1408, 0,
 0.10676, 0, 0.0856006, 0, 0.0712249, 0, 0.0608478, 0,
 0.0530194, 0, 0.0469124, 0, 0.0420211, 0, 0.0380191,
 0, 0.0346867, 0, 0.0318708, 0, 0.0294614, 0,
 0.0273773, 0, 0.0255576, 0, 0.0239557, 0, 0.0225352, 0,
 0.0212672, 0, 0.0201288, 0, 0.0191014, 0, 0.0181696}
```

In[60]:=
```
Clear[cc2];

cc2[n_] = 2/(lL2 Sinh[n Pi/lL2 lK2])
   Integrate[g22[y] Sin[n Pi/lL2 y], {y, 0, lL2}]
```

Out[61]= 0

In[62]:=
```
Clear[dd2];

dd2[n_] = 2/lL2 Integrate[g12[y] Sin[n Pi/lL2 y], {y, 0, lL2}]
```

Out[63]= 0

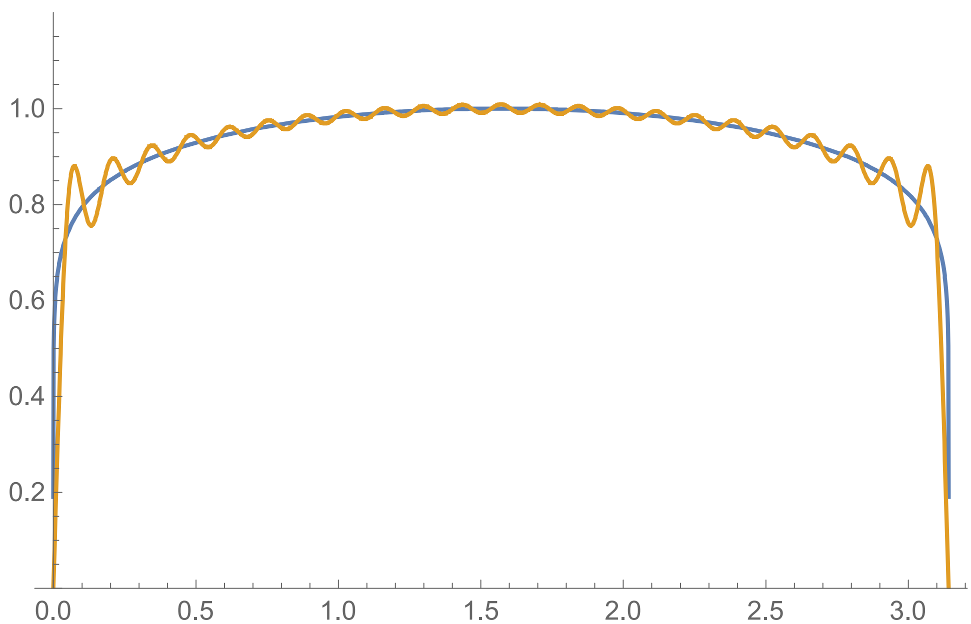
The solution is

In[64]:=
```
Clear[uu2];
```

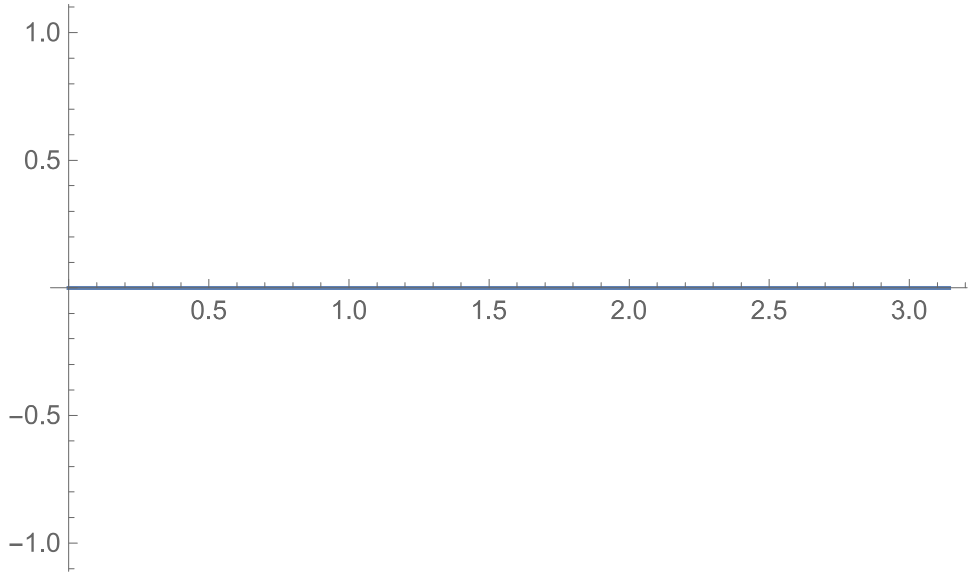
```
uu2[x_, y_] = Sum[aa2[n] Sin[n Pi/lK2 x] Sinh[n Pi/lK2 y]/Sinh[n Pi/lK2 lL2], {n, 1, nn2}] +
  Sum[bbl2[[n]] Sin[n Pi/lK2 x] Sinh[n Pi/lK2 (lL2 - y)]/Sinh[n Pi/lK2 lL2], {n, 1, nn2}] +
  Sum[cc2[n] Sin[n Pi/lL2 y] Sinh[n Pi/lL2 x]/Sinh[n Pi/lL2 lK2], {n, 1, nn2}] +
  Sum[dd2[n] Sin[n Pi/lL2 y] Sinh[n Pi/lL2 (lK2 - x)]/Sinh[n Pi/lL2 lK2], {n, 1, nn2}];
```

In[66]:=
```
Plot[{f12[x], uu2[x, 0]}, {x, 0, lK2}, PlotRange → {0, 1.2}]
```

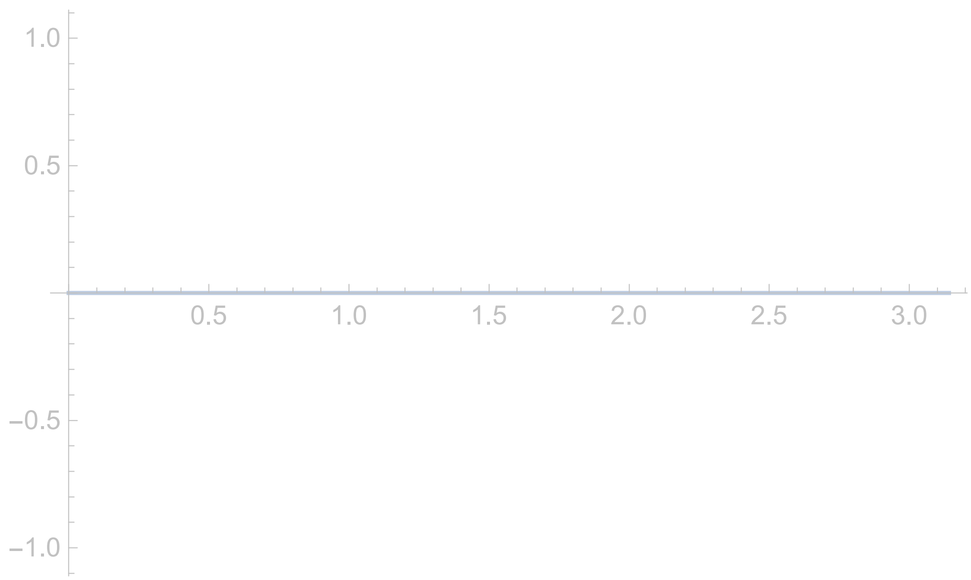
Out[66]=



In[67]:=
```
Plot[uu2[x, lL2], {x, 0, lK2}]
```

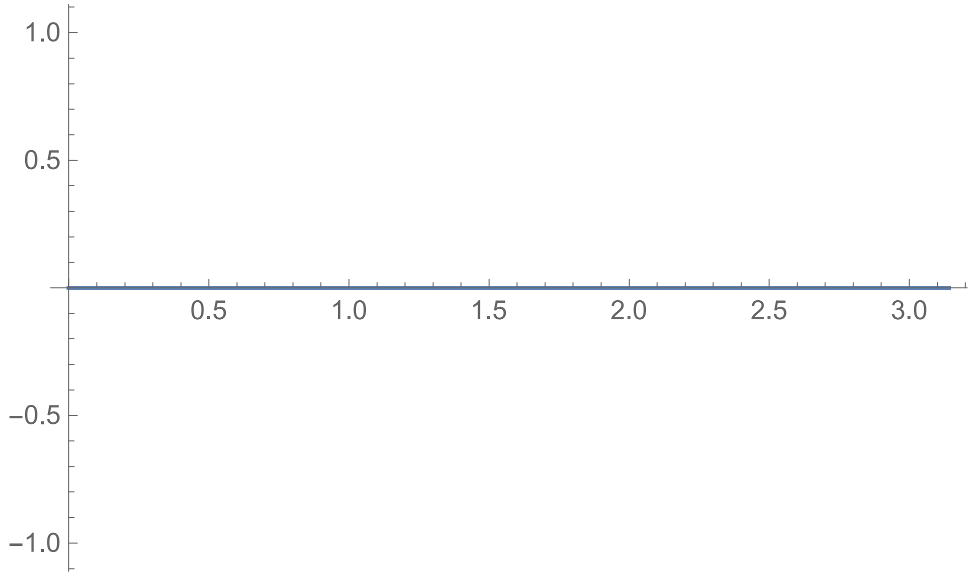
Out[67]=



```
Plot[uu2[0, y], {y, 0, lL2}]
```

Out[68]=


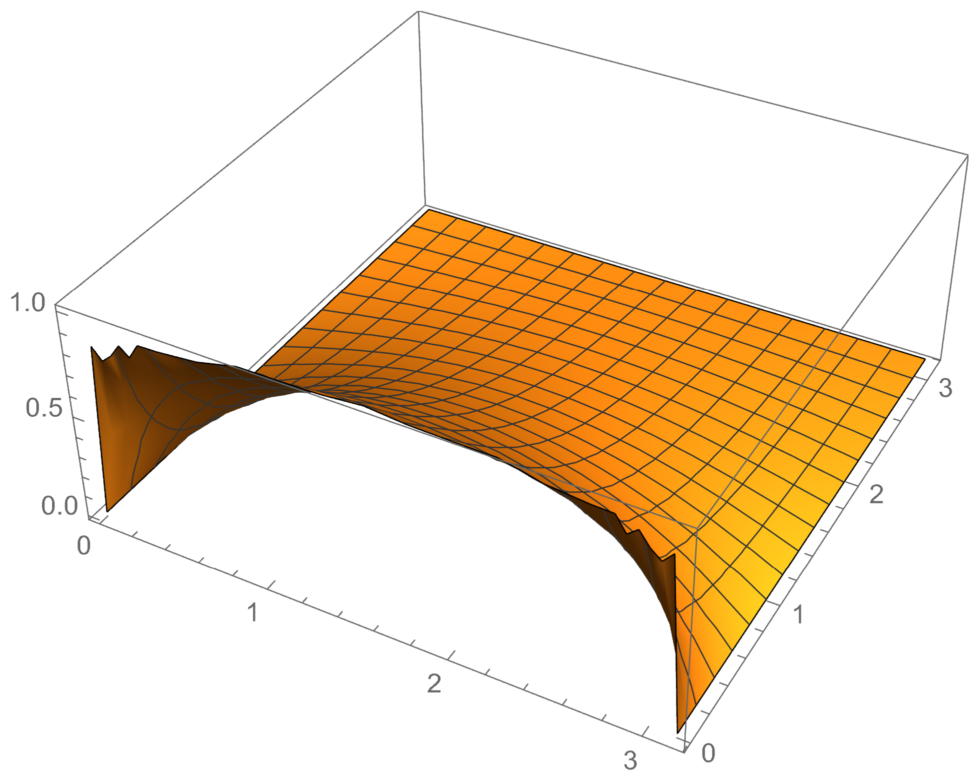
In[70]:= Plot[uu2[lK2, y], {x, 0, lL2}]

Out[70]=



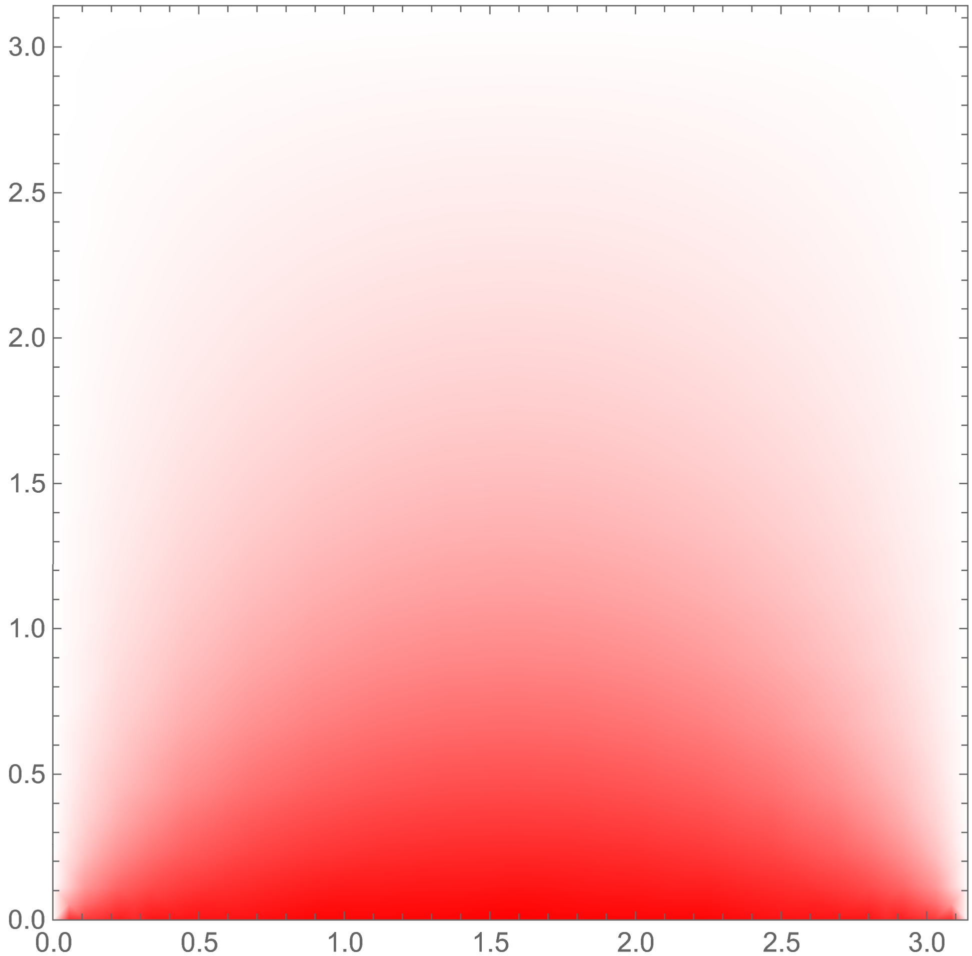
In[71]:= Plot3D[N[uu2[x, y]], {x, 0, lK2}, {y, 0, lL2}]

Out[71]=



In[72]:=
```
DensityPlot[N[uu2[x, y]], {x, 0, lK2}, {y, 0, lL2},
 Frame → True, PlotRange → {{0, lK2}, {0, lL2}},
 ColorFunction → (RGBColor[1, 1 - #, 1 - #] &)]
```

Out[72]=



A symbolic implementation with a problem

Here are the given quantities

In[73]:=
```
Clear[lK3, lL3, f13, f23, g13, g23, nn3];

nn3 = 20;

lK3 = 1; lL3 = 1;

f13[x_] = 3 x^2 + 1;

g23[y_] = 4 - 8 y (y - 1);

f23[x_] = 4 + 8 x (x - 1);

g13[y_] = 1 + 3 y^2;
```

In[80]:=
```
Clear[aa3];

aa3[n_] =
 FullSimplify[
  2/lK3 Integrate[f23[x] Sin[n Pi/lK3 x], {x, 0, lK3}],
  And[n ∈ Integers, n > 0]]
```

Out[81]= $-\frac{8\left(-1+(-1)^{n}\right)\left(-4+n^{2}\pi^{2}\right)}{n^{3}\pi^{3}}$

In[82]:=
```
Clear[bb3];

bb3[n_] =
 FullSimplify[
  2/lK3 Integrate[f13[x] Sin[n Pi/lK3 x], {x, 0, lK3}],
  And[n ∈ Integers, n > 0]]
```

Out[83]= $$\frac{2\left(6\left(-1+(-1)^n\right)+\left(1-4(-1)^n\right)n^2\pi^2\right)}{n^3\pi^3}$$

In[84]:=
```
Clear[cc3];

cc3[n_] =
 FullSimplify[
  2/lL3 Integrate[g23[y] Sin[n Pi/lL3 y], {y, 0, lL3}],
  And[n ∈ Integers, n > 0]]
```

Out[85]= $$-\frac{8\left(-1+(-1)^n\right)\left(4+n^2\pi^2\right)}{n^3\pi^3}$$

In[86]:=
```
Clear[dd3];

dd3[n_] = 2/lL3 Integrate[g13[y] Sin[n Pi/lL3 y], {y, 0, lL3}]
```

Out[87]= $$\frac{2\left(-6+n^2\pi^2+\left(6-4n^2\pi^2\right)\mathrm{Cos}[n\pi]+6n\pi\,\mathrm{Sin}[n\pi]\right)}{n^3\pi^3}$$

The solution is

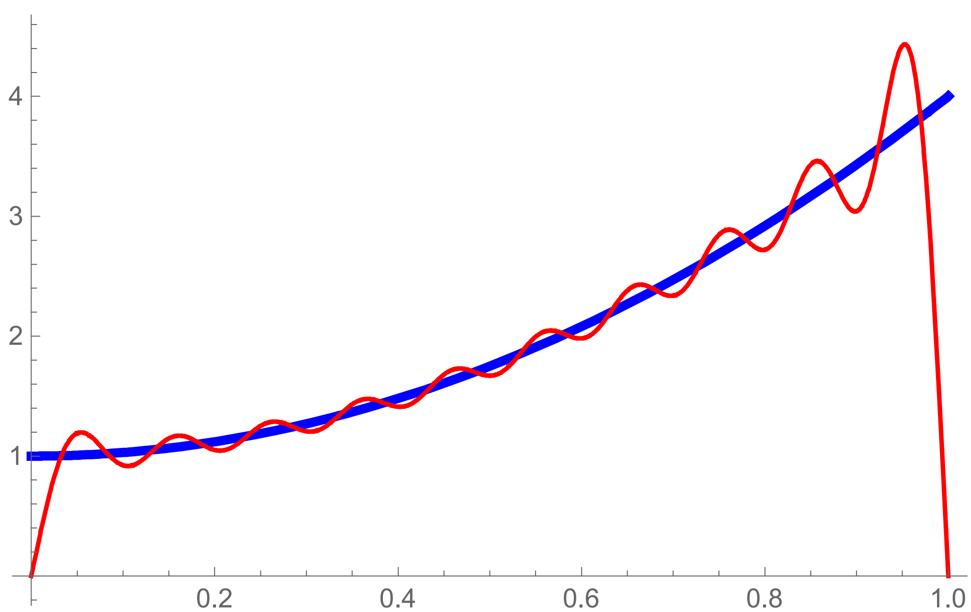
In[88]:=
```
Clear[uu3];
```

$$\text{uu3}[x\_, y\_] = \sum_{n=1}^{\text{nn3}} \text{aa3}[n] \,\text{Sin}\left[\frac{n\,\text{Pi}}{\text{lK3}} x\right] \frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{\text{lK3}} y\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{\text{lK3}} \text{lL3}\right]} + \sum_{n=1}^{\text{nn3}} \text{bb3}[n] \,\text{Sin}\left[\frac{n\,\text{Pi}}{\text{lK3}} x\right] \frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{\text{lK3}} (\text{lL3} - y)\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{\text{lK3}} \text{lL3}\right]} + \sum_{n=1}^{\text{nn3}} \text{cc3}[n] \,\text{Sin}\left[\frac{n\,\text{Pi}}{\text{lL3}} y\right] \frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{\text{lL3}} x\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{\text{lL3}} \text{lK3}\right]} + \sum_{n=1}^{\text{nn3}} \text{dd3}[n] \,\text{Sin}\left[\frac{n\,\text{Pi}}{\text{lL3}} y\right] \frac{\text{Sinh}\left[\frac{n\,\text{Pi}}{\text{lL3}} (\text{lK3} - x)\right]}{\text{Sinh}\left[\frac{n\,\text{Pi}}{\text{lL3}} \text{lK3}\right]};$$

How good are the approximations? Here are visual answers:

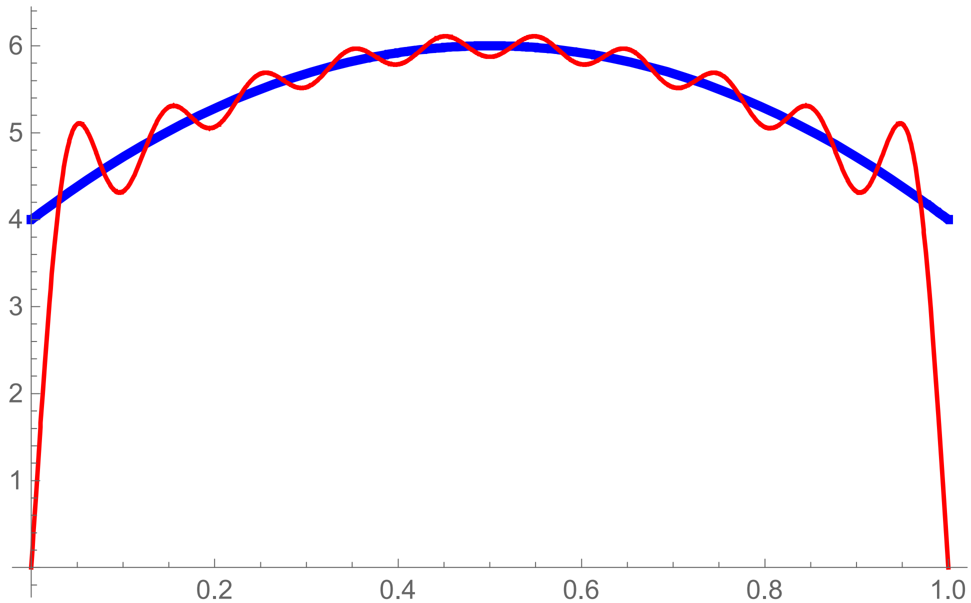
In[90]:=
```
Plot[{f13[x], uu3[x, 0]}, {x, 0, lK3},
 PlotStyle → {{Blue, Thickness[0.01]},
   {Red, Thickness[0.005]}}, PlotRange → All]
```

Out[90]=



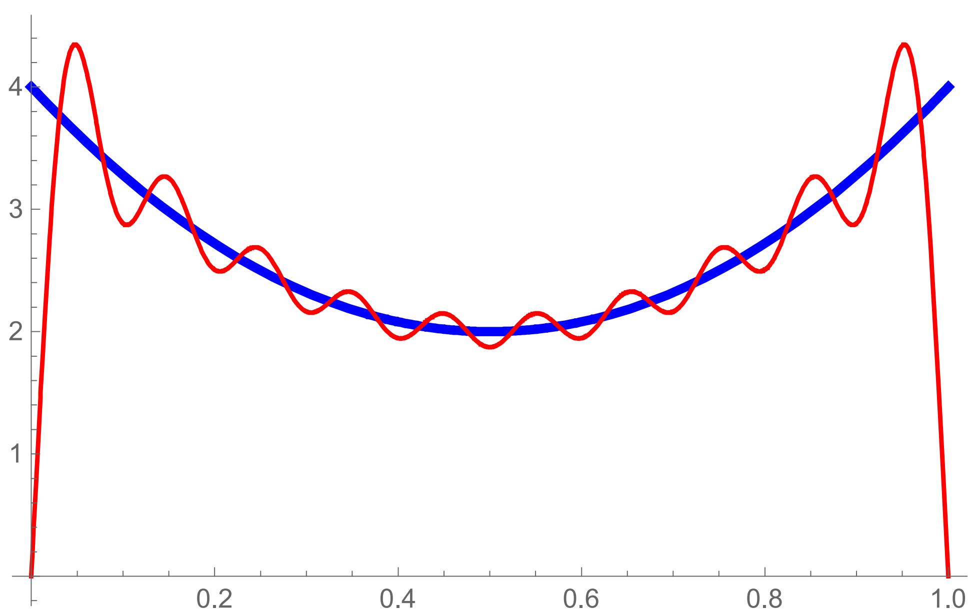
In[91]:=
```
Plot[{g23[y], uu3[lK3, y]}, {y, 0, lL3},
  PlotStyle → {{Blue, Thickness[0.01]},
    {Red, Thickness[0.005]}}, PlotRange → All]
```

Out[91]=



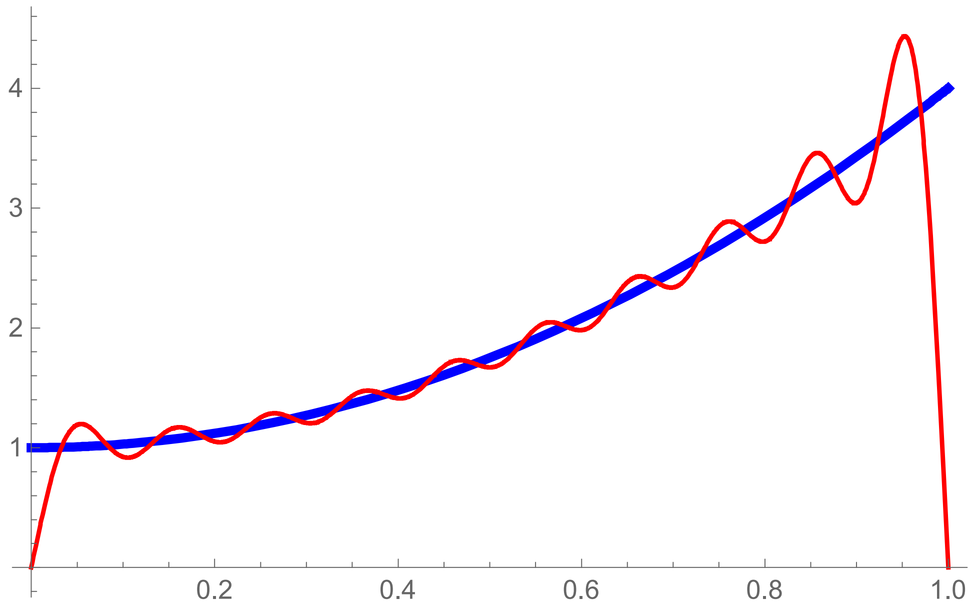
In[92]:=
```
Plot[{f23[x], uu3[x, lL3]}, {x, 0, lK3},
  PlotStyle → {{Blue, Thickness[0.01]},
    {Red, Thickness[0.005]}}, PlotRange → All]
```

Out[92]=



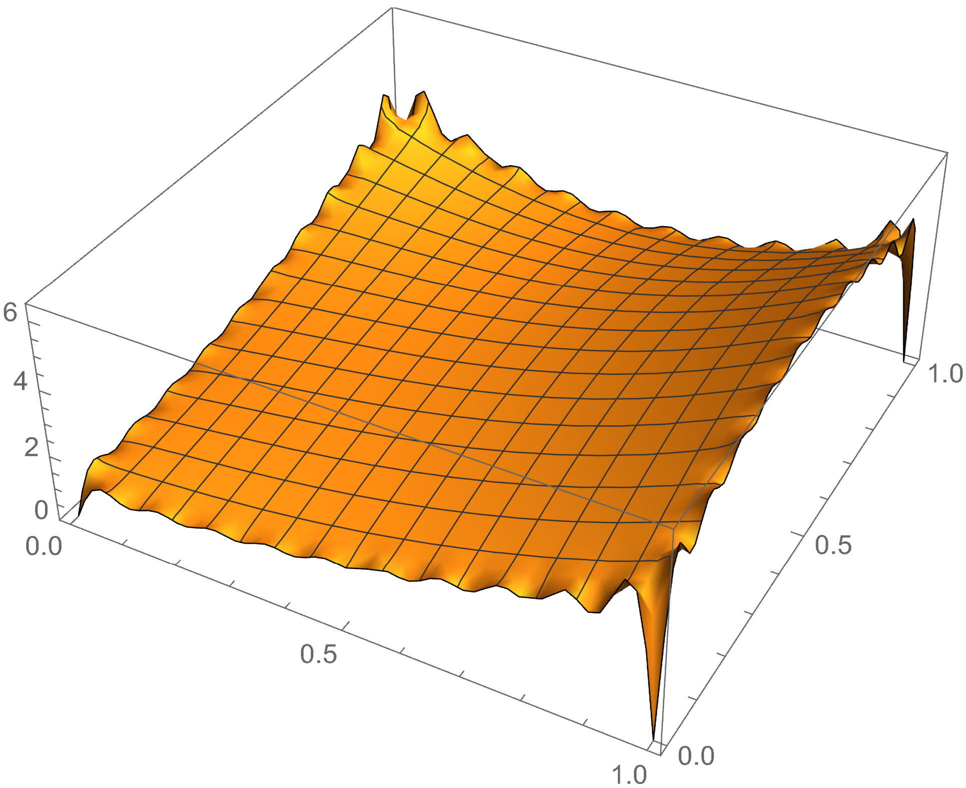
In[93]:=
```
Plot[{g13[y], uu3[0, y]}, {y, 0, lL3},
 PlotStyle → {{Blue, Thickness[0.01]},
   {Red, Thickness[0.005]}}, PlotRange → All]
```

Out[93]=



In[94]:=
```
Plot3D[N[uu3[x, y]], {x, 0, 1K3}, {y, 0, 1L3},
  Mesh → Automatic, PlotRange → {0, 6.5}]
```

Out[94]=



In[95]:=
```
DensityPlot[N[uu3[x, y]], {x, 0, lK3}, {y, 0, lL3},
 Frame → True, PlotRange → {{0, lK3}, {0, lL3}},
 ColorFunction → (RGBColor[1, 1 - #, 1 - #] &)]
```

Out[95]=