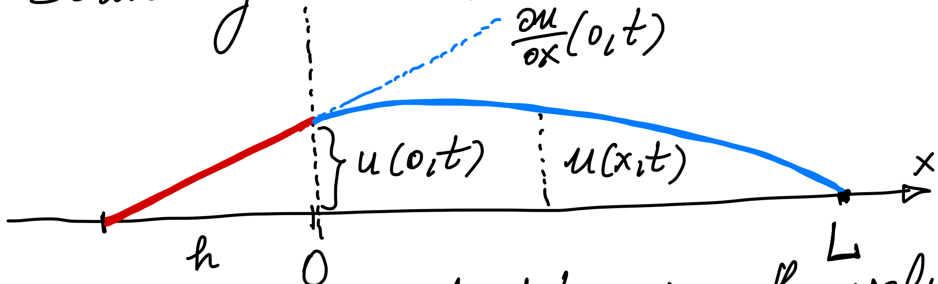
Vibrating String whose Left-end is soaked in super-glue

Robin Boundary Conditions

Consider a vibrating string whose left-end of length h is soaked in super-glue, so it is not flexible. This physical situation leads to Robin Boundry conditions at 0.



The red super-glued part determines the value of the slope of the string at 0, that is $\frac{\partial u}{\partial x}(0,t)$:

$$\frac{\partial u}{\partial x}(0,t) = \frac{u(0,t)}{h}$$

This is a Robin Boundary Condition:

$$u(0,t) - h\frac{\partial u}{\partial x}(0,t) = 0.$$

We consider a uniform string with tension force T_0, mass density ρ_0 and we set $c = \sqrt{T_0/\rho_0}$. In this case, the vibrating string problem is: With $L>0$, $h \in \mathbb{R}\setminus\{0\}$

PDE: $\quad \frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t), \; x \in [0,L], t \in [0,+\infty)$

BCs: $\quad u(0,t) - h\frac{\partial u}{\partial x}(0,t) = 0, u(L,t) = 0, \forall t \in [0,+\infty)$

ICs: $\quad u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x), \; \forall x \in [0,L]$

Here we excluded $h=0$ since in this case we have a Dirichlet Boundary condition which we considered earlier. (Although this case will be implicitly included in our considerations.)

Here f and g are piecewise smooth functions such that

$$f(0) - hf'(0) = 0 \text{ and } f(L) = 0$$

(no boundary conditions on g)

Separation of variables $u(x,t) = A(x)B(t)$ leads to the second order equation for B:

$$B''(t) = -\lambda c^2 B(t)$$

and the boundary-eigenvalue problem for A:

$$A''(x) = -\lambda A(x)$$

$$A(0) - hA'(0) = 0$$

$$A(L) = 0$$

It will be proved for an arbitrary Sturm-Liouville problem that such a problem does not have non-real eigenvalues.

So we consider three possibilities for λ:

Case 1. $\lambda < 0$ Case 2. $\lambda = 0$ Case 3. $\lambda > 0$

Case 1. $\lambda < 0$. We set $\lambda = -\mu^2$ with $\mu > 0$.

We need to find those values for μ for which there exist a non-zero function $A(x)$ such that:

$$A''(x) = \mu^2 A(x) \text{ and } A(0) - hA'(0) = 0 \text{ and } A(L) = 0.$$

The reasoning in problems like this is based on the fact that we know the fundamental set of solutions of

$$A''(x) = \mu^2 A(x).$$

In this case we chose to work with the fundamental set of solutions $\cosh(\mu x)$ and $\sinh(\mu x)$, which we abbreviate as $ch(\mu x)$ and $sh(\mu x)$: We need to find (nonzero)

$$A(x) = C_1\, ch(\mu x) + C_2\, sh(\mu x)$$

which satisfies the boundary conditions

a big step in right direction and function A is converted to red C_1, C_2

BCs: $A(0) - h A'(0) = 0$ and $A(L) = 0$.

So, the question is: For which $\mu > 0$ there exist nonzero $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ such that $C_1 \, ch(\mu x) + C_2 \, sh(\mu x)$ satisfies the above BCs.

To answer this question we substitute into BCs and consider the linear system that we obtain:

$$A(0) = C_1, \quad A'(0) = \mu C_2, \quad A(L) = C_1 \, ch(\mu L) + C_2 \, sh(\mu L)$$

The system is:

$$C_1 - h \mu C_2 = 0$$

$$C_1 \, ch(\mu L) + C_2 \, sh(\mu L) = 0$$

Written as a matrix equation this system reads

$$\begin{bmatrix} 1 & -h\mu \\ \mathrm{ch}(\mu L) & \mathrm{sh}(\mu L) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The logic here is similar to finding eigenvalues of a 2×2 matrix. The difficulty here is that μ is involved in ch and sh.

The above matrix equation has a nontrivial solution $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ if and only if

$$\begin{vmatrix} 1 & -h\mu \\ \mathrm{ch}(\mu L) & \mathrm{sh}(\mu L) \end{vmatrix} = 0,$$

that is

$$\mathrm{sh}(\mu L) + h\mu\, \mathrm{ch}(\mu L) = 0.$$

It is not likely that we can find a symbolic solution of this equation. So, let us solve it "graphically":

$\mathrm{ch}(\mu L) > 0$, so
$$\underbrace{\frac{\mathrm{sh}(\mu L)}{\mathrm{ch}(\mu L)}}_{\mathrm{th}(\mu L)} = -h\mu$$

So we need to understand the solutions of the equation
$$\mathrm{th}(\mu L) = -h\mu .$$

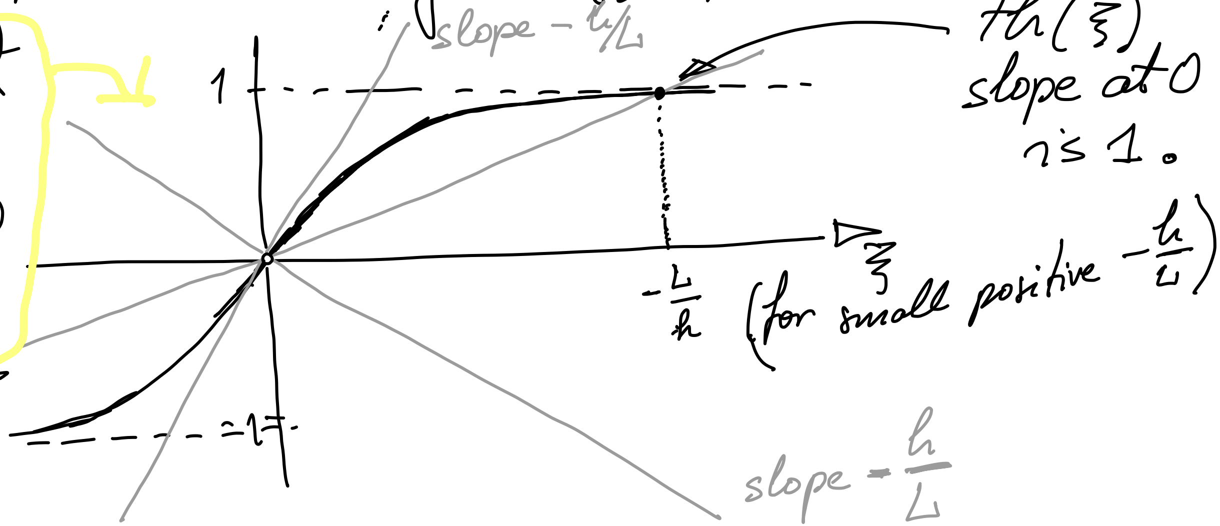
It is easier to picture the solutions of this equation if we introduce a new variable $\xi = \mu L$ and study

$\xi > 0$ since $\mu > 0$

$$\mathrm{th}(\xi) = -\frac{h}{L}\xi .$$

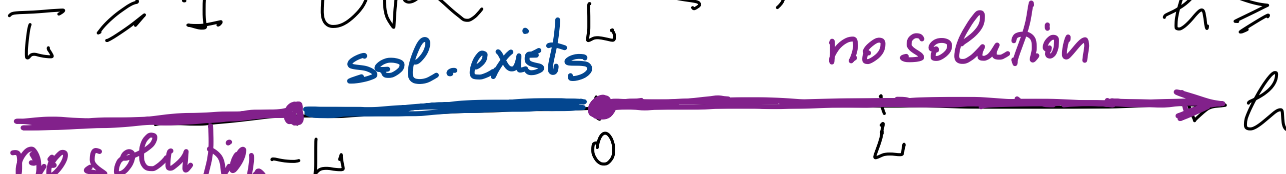
Now we can graph $\mathrm{th}(\xi)$ and the line through the origin and analyze the existence of a solution:

we are not interested in $\xi \leq 0$ since $\mu > 0$.



What we see from the above graph is that the equation $th(\xi) = -\frac{h}{L}\xi$ has no solution if

$-\frac{h}{L} \geq 1$ OR $-\frac{h}{L} \leq 0$, that is $-h \geq L$ or $h \geq 0$



Thus the solution exists if $-L < h < 0$.

If $-L < h < 0$ is satisfied, then Mathematica can solve the equation

$$\operatorname{th}(\mu L) = -h\mu .$$

in Mathematica

We use the command FindRoot[,].

Notice that if $-\frac{h}{L}$ is a small positive number, then a good approximation for the solution for μ is $-\frac{1}{h}$.

comment.

Denote the solution found by Mathematica by μ_{-1}. Then $\lambda_{-1} = -(\mu_{-1})^2$.

Now we go back to the matrix equation that we need to solve:

$$\begin{bmatrix} 1 & -h\mu_{-1} \\ ch(\mu_{-1}L) & sh(\mu_{-1}L) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

With the μ_{-1} that we found the 2×2 matrix is singular and since we need only one pair $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ we can choose $c_1 = h\mu_{-1}$ and $c_2 = 1$

Hence the eigenvalue (negative one) is $\lambda_{-1} = -(\mu_{-1})^2$ and a corresponding eigenfunction is:

$$h\mu_{-1}\,ch(\mu_{-1}x) + sh(\mu_{-1}x)$$

Now we go back and solve the equation for $B(t)$:

$$B''(t) = (\mu_{-1})^2 c^2 B(t).$$

The general solution of this equation is:

$$a_{-1}\,ch(\mu_{-1}ct) + b_{-1}\,sh(\mu_{-1}ct)$$

where a_{-1} and b_{-1} are arbitrary constants.

Finally we write two separated solutions

$$ch(\mu_{-1}ct)\left(h\mu_{-1}\,ch(\mu_{-1}x) + sh(\mu_{-1}x)\right)$$

and $$sh(\mu_{-1}ct)\left(h\mu_{-1}\,ch(\mu_{-1}t) + sh(\mu_{-1}t)\right)$$

These two solutions correspond to "natural modes" of vibrations; the only difference being that these solutions do not vibrate; practically they mean the string will break.

Case 2 $\lambda = 0$. We need to answer the question:

Is there a function $A(x)$ such that

$$A''(x) = 0$$

and BCs $A(0) - h A'(0) = 0$ and $A(L) = 0$.

A fundamental set of solutions of $A''(x) = 0$ is $\{1, x\}$.

The general solution is $A(x) = C_1 + C_2 x$.

Substitude into BCs:

$$C_1 - h C_2 = 0$$

$$C_1 + L C_2 = 0$$

Written as a matrix equation:

$$\begin{bmatrix} 1 & -h \\ 1 & L \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The preceding matrix equation has a nontrivial solution if and only if $\begin{vmatrix} 1 & -h \\ 1 & L \end{vmatrix} = 0$, that is $L + h = 0$.

Hence $\lambda = 0$ is an eigenvalue if and only if $h = -L$.

In this case a corresponding eigenfunction is

$L - x$ (here $c_1 = L$, $c_2 = -1$)

Now solve the time equation $B''(t) = 0$.
The general solution is $a_0 \cdot 1 + b_0 t$.
Thus, two special "separated" solutions of the PDE and the BCs are

$$1(L-x) \quad \text{and} \quad t(L-x)$$

these solution are relevant only in the case $h = -L$.
Whenever $h \neq -L$, 0 is <u>NOT</u> an eigenvalue.

<u>Case 3</u> $\lambda > 0$. Set $\lambda = \mu^2$ with $\mu > 0$.
We need to find all positive values of μ for which there exists a nonzero function $A(x)$ such

that $$A''(x) = -\mu^2 A(x)$$

and BCs $A(0) - h A'(0) = 0 \quad A(L) = 0.$

The fundamental set of solutions is $\{\cos(\mu x), \sin(\mu x)\}$.

the general solution is $A(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$

This expression is a big step in right direction.
The unknown function $A(x)$ is replaced by two unknow numbers C_1 and C_2; certainly an easier task to find.

Calculate: $A(0) = C_1$, $A'(0) = \mu C_2$

$$A(L) = C_1 c(\mu L) + C_2 s(\mu L)$$

and substitute into boundry conditions:

$$C_1 - h\mu C_2 = 0$$

$$c(\mu L)C_1 + s(\mu L)C_2 = 0$$

Write it as a matrix equation:

$$\begin{bmatrix} 1 & -h\mu \\ c(\mu L) & s(\mu L) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This matrix equation has a nontrivial solution if and only if

$$s(\mu L) + h\mu c(\mu L) = 0. \quad (*)$$

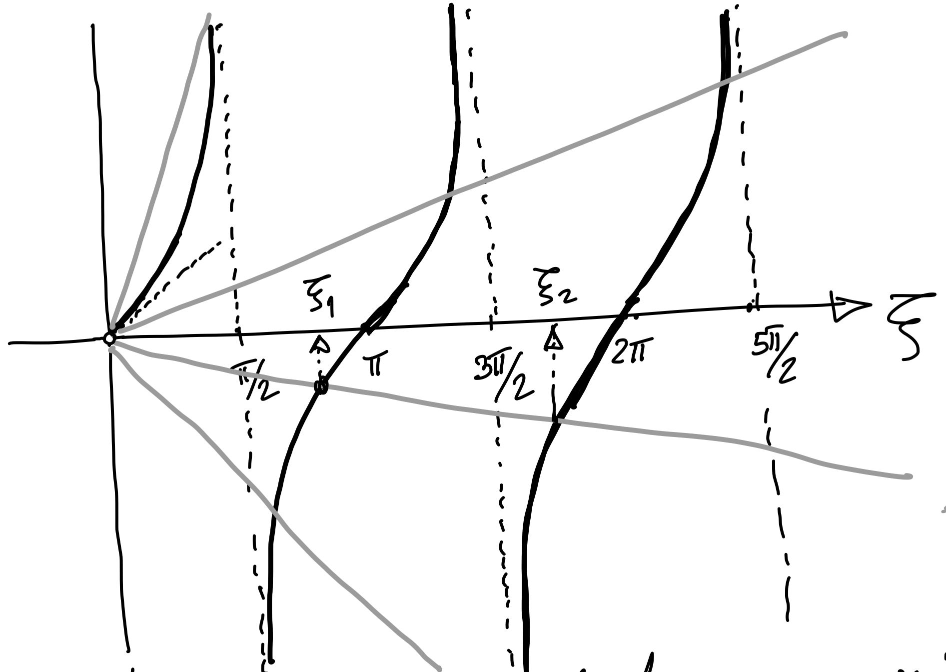
It is desirable to reduce turning two two places. We can achieve that by dividing by $c(\mu L)$. For that

we need $c(\mu L) \neq 0$. So, we first consider the case $\mu L = (2k-1)\frac{\pi}{2}$ with $k \in \mathbb{N}$. In this case $\cos(\mu L) = 0$ and $\sin(\mu L) = (-1)^{k+1}$.
In this case $(*)$ becomes $(-1)^{k+1} + 0 = 0$ which is never true. Therefore we will lose no solutions if we assume $c(\mu L) \neq 0$. Then $(*)$ is equivalent to

$$\tan(\mu L) = -h\mu.$$

Introducing $\xi = \mu L$ we look for the positive solutions of

$$\underbrace{\tan(\xi)}_{\text{black graph}} = \underbrace{-\frac{h}{L}\xi}_{\text{gray line}}.$$



There are countably many solutions regardless of the choice of h and L.

(in this case $h > 0$. In fact $h \simeq \frac{L}{2}$.)

Mathematica can calculate approximations for $\mu_1, \mu_2, \mu_3, \dots, \mu_n, \dots$. In fact, when calculating these approximations I use the equation (*)

$$\sin(\mu L) + \mu L \cos(\mu L) = 0.$$

Now we can consider $\mu_1, \mu_2, \mu_3, \dots, \mu_n, \dots$ green. We know them (to some extend). What are the corresponding eigenfunctions? The eigenfunction corresponding to the eigenvalue $\lambda_n = (\mu_n)^2$ is

$$\mu_n L \cdot \cos(\mu_n x) + \sin(\mu_n x)$$

To find the natural modes of vibrations we need to solve $B''(t) = -(\mu_n)^2 c^2 B(t)$

the fundamental set of solutions is

$$\cos(\mu_n c t), \quad \sin(\mu_n c t)$$

A typical natural mode of vibration is

$$\cos(\mu_n c t)\left(\mu_n h \cdot \cos(\mu_n x) + \sin(\mu_n x)\right)$$

We can ignore other forms (of nat. modes) since they are just shifts and scales of this one. The End!

(time shifts)

or, I better continue and

Find the solution that satisfies the initial conditions. Recall we are solving:

PDE : $\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad x \in [0,L], \quad t \in [0,+\infty)$

BCs : $u(0,t) - h\frac{\partial u}{\partial x}(0,t) = 0, \quad u(L,t) = 0, \quad \forall t \in [0,+\infty)$

ICs : $u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad \forall x \in [0,L]$

where $L > 0$, $c > 0$, $h \in \mathbb{R}$ are given real numbers and $f : [0,L] \to \mathbb{R}$, $g : [0,L] \to \mathbb{R}$ are given functions, and f is such that

$$f(0) - hf'(0) = 0 \quad \text{and} \quad f(L) = 0.$$

We have found the natural modes of vibrations of this string. They are :

$$\left(\cos(\mu_n c t)\right)\left(\mu_n h \cdot \cos(\mu_n x) + \sin(\mu_n x)\right)$$

and

$$\left(\sin(\mu_n c t)\right)\left(\mu_n h \cdot \cos(\mu_n x) + \sin(\mu_n x)\right)$$

The natural modes of vibration are the separated solutions of the PDE + BCs. Since both PDE and BCs are homogeneous, the superposition principle holds. Thus the linear combinations of the natural modes of vibration are also solutions of PDE + BCs.

We set

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\mu_n c t)\, \varphi_n(x) + \sum_{n=1}^{\infty} b_n \sin(\mu_n c t)\, \varphi_n(x),$$

where we set $\varphi_n(x) = \mu_n h \cdot \cos(\mu_n x) + \sin(\mu_n x)$, $x \in [0, L]$.

To satisfy the Initial Conditions we need $\frac{\partial u}{\partial t}(x,t)$:

$$\frac{\partial u}{\partial t}(x,t) = -\sum_{n=1}^{\infty} a_n \mu_n c \sin(\mu_n c t)\, \varphi_n(x) + \sum_{n=1}^{\infty} b_n \mu_n c \cos(\mu_n c t)\, \varphi_n(x)$$

Substitute the infinite sums expressions for $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$ in the Initial Conditions to get:

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} b_n \mu_n c\, \varphi_n(x),$$

$x \in [0, L]$.

An important property of the eigenfunctions $\varphi_n(x)$ is that they are orthogonal with respect to the inner product

$$\langle u, v \rangle = \int_0^L u(x)\, v(x)\, dx.$$

that is

$$\forall m, n \in \mathbb{N} \text{ we have } \int_0^L \varphi_m(x)\, \varphi_n(x)\, dx = 0$$

whenever $m \neq n$.

Since the functions $\varphi_n(x)$ are nonzero functions, we have

$$\int_0^L (\varphi_n(x))^2 dx > 0, \quad \forall n \in \mathbb{N}.$$

We use these two properties to calculate a_n-s and b_n-s using f and g.

Recall

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} b_n \mu_n c\, \varphi_n(x), \quad x \in [0, L].$$

Let $k \in \mathbb{N}$ be arbitrary and multiply both above equalities by $\varphi_k(x)$. We get:

$$f(x)\varphi_k(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)\varphi_k(x) \quad \text{and} \quad g(x)\varphi_k(x) = \sum_{n=1}^{\infty} b_n \mu_n c\, \varphi_n(x)\varphi_k(x).$$

Integrate the above equalities from 0 to L, to get

$$\int_0^L f(x)\varphi_k(x)dx = \sum_{n=1}^{\infty} a_n \int_0^L \varphi_n(x)\varphi_k(x)dx \quad \text{and} \quad \int_0^L g(x)\varphi_k(x)dx = \sum_{n=1}^{\infty} b_n \mu_n c \int_0^L \varphi_n(x)\varphi_k(x)dx.$$

Now recall the orthogonality: $\int_0^L \varphi_n(x)\varphi_k(x)dx = 0$ for $n \neq k$.

Therefore:

$$\int_0^L f(x)\varphi_k(x)\,dx = a_k\int_0^L(\varphi_k(x))^2\,dx$$

and
$$\int_0^L g(x)\varphi_k(x)\,dx = b_k\mu_k c\int_0^L(\varphi_k(x))^2\,dx.$$

Finally we leave the formulas for a_k and b_k.

For all $k\in\mathbb{N}$

$$a_k = \frac{\int_0^L f(x)\varphi_k(x)\,dx}{\int_0^L(\varphi_k(x))^2\,dx} \quad \text{and} \quad b_k = \frac{\int_0^L g(x)\varphi_k(x)\,dx}{\mu_k c\int_0^L(\varphi_k(x))^2\,dx}$$

above formulas greenify a_k and b_k.

With these a_k and b_k (being greenified) the solution of PDE + BCs + ICs is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\mu_n c t)\,\varphi_n(x) + \sum_{n=1}^{\infty} b_n \sin(\mu_n c t)\,\varphi_n(x).$$

These formulas are beautifully implemented in Mathematica. We need not work with infinite series, it is sufficient to use twenty, or so, terms in the series to get good approximation for the solution.