Inner Product Spaces

Branko Ćurgus

March 14, 2023 at 01:44

1 Inner Product Spaces

We will first introduce several "dot-product-like" objects. We start with the most general.

Definition 1.1. Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$. A function

$$[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$$

is a *sesquilinear form* on $\mathscr{V}$ if the following two conditions are satisfied.

(a) (linearity in the first variable)

$$\forall\, \alpha, \beta \in \mathbb{F} \quad \forall\, u, v, w \in \mathscr{V} \quad [\alpha u + \beta v, w] = \alpha[u, w] + \beta[v, w].$$

(b) (anti-linearity in the second variable)

$$\forall\, \alpha, \beta \in \mathbb{F} \quad \forall\, u, v, w \in \mathscr{V} \quad [u, \alpha v + \beta w] = \overline{\alpha}[u, v] + \overline{\beta}[u, w].$$

ex-sfnh **Example 1.2.** Let $M \in \mathbb{C}^{n \times n}$ be arbitrary. Then

$$[\mathbf{x}, \mathbf{y}] = (M\mathbf{x}) \cdot \mathbf{y}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{C}^n,$$

is a sesquilinear form on the complex vector space $\mathbb{C}^n$. Here $\cdot$ denotes the usual dot product in $\mathbb{C}$.

An abstract form of the Pythagorean Theorem holds for sesquilinear forms.

Theorem 1.3 (Pythagorean Theorem)**.** *Let $[\,\cdot\,,\cdot\,]$ be a sesquilinear form on a vector space $\mathscr{V}$ over a scalar field $\mathbb{F}$. If $v_1, \cdots, v_n \in \mathscr{V}$ are such that $[v_j, v_k] = 0$ whenever $j \neq k, j, k \in \{1, \dots, n\}$, then*

$$\left[\sum_{j=1}^{n} v_j, \sum_{k=1}^{n} v_k\right] = \sum_{j=1}^{n} [v_j, v_j].$$

Proof. Assume that $[v_j, v_k] = 0$ whenever $j \neq k, j, k \in \{1, \dots, n\}$ and apply the additivity of the sesquilinear form in both variables to get:

$$\left[\sum_{j=1}^{n} v_j, \sum_{k=1}^{n} v_k\right] = \sum_{j=1}^{n}\sum_{k=1}^{n}[v_j, v_k] = \sum_{j=1}^{n}[v_j, v_j]. \qquad \square$$

th-poli **Theorem 1.4** (Polarization identity)**.** *Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$ and let $[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ be a sesquilinear form on $\mathscr{V}$. If $\mathrm{i} \in \mathbb{F}$, then*

$$[u, v] = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \big[u + \mathrm{i}^k v, u + \mathrm{i}^k v\big] \qquad (1)$$ eq-pi

for all $u, v \in \mathscr{V}$.

Proof. For the proof we expend the sum on the right hand side, ignoring the fraction 1/4, using the linearity in the first variable and anti-linearity in the second variable. The resulting expression will have the following four values of the sesquilinear form: $[u, u]$, $[u, v]$, $[v, u]$, $[v, v]$. For each of these values and for each $k \in \{0, 1, 2, 3\}$ we present the corresponding coefficients in a table with the values of the form in the header and values for each k in each row:

	$[u,u]$	$[u,v]$	$[v,u]$	$[v,v]$
$k=0$	1	1	1	1
$k=1$	i	1	-1	i
$k=2$	-1	1	1	-1
$k=3$	$-\mathrm{i}$	1	-1	$-\mathrm{i}$
sum	0	4	0	0

$\square$

co-slf-0 **Corollary 1.5.** *Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$ and let $[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ be a sesquilinear form on $\mathscr{V}$. If $\mathrm{i} \in \mathbb{F}$ and $[v, v] = 0$ for all $v \in \mathscr{V}$, then $[u, v] = 0$ for all $u, v \in \mathscr{V}$.*

Definition 1.6. Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$. A sesquilinear form $[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ is *hermitian* if

(c) (hermiticity) $\quad \forall\, u, v \in \mathscr{V} \quad \overline{[u, v]} = [v, u]$.

A hermitian sesquilinear form is also called an *inner product*.

co-slf-her **Corollary 1.7.** *Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$ such that $\mathrm{i} \in \mathbb{F}$. Let $[\,\cdot\,,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathbb{F}$ be a sesquilinear form on $\mathscr{V}$. Then $[\,\cdot\,,\cdot\,]$ is hermitian if and only if $[v,v] \in \mathbb{R}$ for all $v \in \mathscr{V}$.*

Proof. The "only if" direction follows from the definition of a hermitian sesquilinear form. To prove "if" direction assume that $[v,v] \in \mathbb{R}$ for all $v \in \mathscr{V}$. Let $u, v \in \mathscr{V}$ be arbitrary. We will prove that $\overline{[v,u]} = [u,v]$. By the polarization identity

$$[v,u] = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k \big[v + \mathrm{i}^k u, v + \mathrm{i}^k u\big] \tag{2}$$ eq-vvr

It follows from the assumption that $\big[v + \mathrm{i}^k u, v + \mathrm{i}^k u\big] \in \mathbb{R}$ for all $k \in \{0,1,2,3\}$. Therefore, in (2) we need to conjugate only i^k for $k \in \{0,1,2,3\}$ to calculate $\overline{[v,u]}$. Since $\overline{\mathrm{i}^k} = (-\mathrm{i})^k$, from (2) we obtain

$$\overline{[v,u]} = \frac{1}{4} \sum_{k=0}^{3} (-\mathrm{i})^k \big[v + \mathrm{i}^k u, v + \mathrm{i}^k u\big] \tag{3}$$ eq-vvr2

As in the proof of the Polarization Identity we expend the sum on the right hand side in (3), ignoring the fraction $1/4$, using the linearity in the first variable and anti-linearity in the second variable. The resulting expression will have the following four values of the sesquilinear form: $[u,u]$, $[u,v]$, $[v,u]$, $[v,v]$. For each of these values and for each $k \in \{0,1,2,3\}$ we present the corresponding coefficients in a table with the values of the form in the header and values for each k in each row:

	$[u,u]$	$[u,v]$	$[v,u]$	$[v,v]$
$k=0$	1	1	1	1
$k=1$	$-\mathrm{i}$	1	-1	$-\mathrm{i}$
$k=2$	-1	1	1	-1
$k=3$	i	1	-1	i
sum	0	4	0	0

Hence, the sum in (3) is identical to the sum in (1). Therefore $\overline{[v,u]} = [u,v]$. ◻

Let $[\,\cdot\,,\cdot\,]$ be an inner product on $\mathscr{V}$. The hermiticity of $[\,\cdot\,,\cdot\,]$ implies that $\overline{[v,v]} = [v,v]$ for all $v \in \mathscr{V}$. Thus $[v,v] \in \mathbb{R}$ for all $v \in \mathscr{V}$. The natural trichotomy that arises is the motivation for the following definition.

Definition 1.8. An inner product $[\,\cdot\,,\cdot\,]$ on $\mathscr{V}$ is called *nonnegative* if $[v,v] \geq 0$ for all $v \in \mathscr{V}$, it is called *nonpositive* if $[v,v] \leq 0$ for all $v \in \mathscr{V}$, and it is called *indefinite* if there exist $u \in \mathscr{V}$ and $v \in \mathscr{V}$ such that $[u,u] < 0$ and $[v,v] > 0$.

2 Nonnegative inner products

The following implication that you might have learned in high school will be useful below.

Theorem 2.1 (High School Theorem)**.** *Let a, b, c be real numbers. Assume $a \geq 0$. Then the following implication holds:*

$$\forall\, x \in \mathbb{Q} \quad ax^2 + bx + c \geq 0 \qquad \Rightarrow \qquad b^2 - 4ac \leq 0. \tag{4}$$

eq-impl

Theorem 2.2 (Cauchy-Bunyakovsky-Schwartz Inequality)**.** *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a nonnegative inner product on $\mathscr{V}$. Then*

$$\forall\, u, v \in \mathscr{V} \quad |\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle. \tag{5}$$

eq-CSBi

The equality occurs in (5) *if and only if there exists $\alpha, \beta \in \mathbb{F}$ not both* 0 *such that $\langle \alpha u + \beta v, \alpha u + \beta v\rangle = 0$.*

Proof. Let $u, v \in \mathscr{V}$ be arbitrary. Since $\langle\cdot,\cdot\rangle$ is nonnegative we have

$$\forall\, t \in \mathbb{Q} \qquad \big\langle u + t\langle u, v\rangle v, u + t\langle u, v\rangle v\big\rangle \geq 0. \tag{6}$$

eq-CSBst1

Since $\langle\cdot,\cdot\rangle$ is a sesquilinear hermitian form on $\mathscr{V}$, (6) is equivalent to

$$\forall\, t \in \mathbb{Q} \qquad \langle u, u\rangle + 2t|\langle u, v\rangle|^2 + t^2|\langle u, v\rangle|^2\langle v, v\rangle \geq 0. \tag{7}$$

eq-CSBst2

As $\langle v, v\rangle \geq 0$, the High School Theorem applies and (7) implies

$$4|\langle u, v\rangle|^4 - 4|\langle u, v\rangle|^2\langle u, u\rangle\langle v, v\rangle \leq 0. \tag{8}$$

eq-CSBst3

Again, since $\langle u, u\rangle \geq 0$ and $\langle v, v\rangle \geq 0$, (8) is equivalent to

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle.$$

Since $u, v \in \mathscr{V}$ were arbitrary, (5) is proved.

Next we prove the claim related to the equality in (5). We first prove the "if" part. Assume that $u, v \in \mathscr{V}$ and $\alpha, \beta \in \mathbb{F}$ are such that $|\alpha|^2 + |\beta|^2 > 0$ and

$$\langle \alpha u + \beta v, \alpha u + \beta v\rangle = 0$$

We need to prove that $|\langle u,v\rangle|^2 = \langle u,u\rangle\langle v,v\rangle$.

Since $|\alpha|^2+|\beta|^2>0$, we have two cases $\alpha \neq 0$ or $\beta \neq 0$. We consider the case $\alpha \neq 0$. The case $\beta \neq 0$ is similar. Set $w = \alpha u + \beta v$. Then $\langle w,w\rangle = 0$ and $u = \gamma v + \delta w$ where $\gamma = -\beta/\alpha$ and $\delta = 1/\alpha$. Notice that the Cauchy-Bunyakovsky-Schwarz inequality and $\langle w,w\rangle = 0$ imply that $\langle w,x\rangle = 0$ for all $x \in \mathscr{V}$. Now we calculate

$$|\langle u,v\rangle| = |\langle \gamma v + \delta w, v\rangle| = |\gamma\langle v,v\rangle + \delta\langle w,v\rangle| = |\gamma\langle v,v\rangle| = |\gamma|\langle v,v\rangle$$

and

$$\langle u,u\rangle = \langle \gamma v + \delta w, \gamma v + \delta w\rangle = \langle \gamma v, \gamma v\rangle = |\gamma|^2\langle v,v\rangle.$$

Thus,

$$|\langle u,v\rangle|^2 = |\gamma|^2\langle v,v\rangle^2 = \langle u,u\rangle\langle v,v\rangle.$$

This completes the proof of the "if" part.

To prove the "only if" part, assume $|\langle u,v\rangle|^2 = \langle u,u\rangle\langle v,v\rangle$. If $\langle v,v\rangle = 0$, then with $\alpha = 0$ and $\beta = 1$ we have

$$\langle \alpha u + \beta v, \alpha u + \beta v\rangle = \langle v,v\rangle = 0.$$

If $\langle v,v\rangle \neq 0$, then with $\alpha = \langle v,v\rangle$ and $\beta = -\langle u,v\rangle$ we have $|\alpha|^2+|\beta|^2>0$ and

$$\langle \alpha u + \beta v, \alpha u + \beta v\rangle = \langle v,v\rangle\big(\langle v,v\rangle\langle u,u\rangle - |\langle u,v\rangle|^2 - |\langle u,v\rangle|^2 + |\langle u,v\rangle|^2\big) = 0.$$

This completes the proof of the characterization of equality in the Cauchy-Bunyakovsky-Schwartz Inequality. □

Corollary 2.3. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a nonnegative inner product on $\mathscr{V}$. Then*

$$\mathscr{N} = \big\{ v \in \mathscr{V} : \langle v,v\rangle = 0 \big\}$$

is a subspace of $\mathscr{V}$.

Proof. Let $v \in \mathscr{N}$, that is let $\langle v,v\rangle = 0$. Let $u \in \mathscr{V}$ be arbitrary. Then by Cauchy-Bunyakovsky-Schwarz inequality we have $|\langle u,v\rangle|^2 \leq \langle u,u\rangle\langle u,u\rangle$. Consequently $\langle u,v\rangle = 0$ for all $u \in \mathscr{V}$. Let now $u,v \in \mathscr{N}$ and $\alpha,\beta \in \mathbb{F}$ be arbitrary. Then

$$\langle \alpha u + \beta v, \alpha u + \beta v\rangle = |\alpha|^2\langle u,u\rangle + \alpha\overline{\beta}\langle u,v\rangle + \overline{\alpha}\beta\langle v,u\rangle + |\beta|^2\langle v,v\rangle = 0,$$

since $\langle u,u\rangle = \langle u,v\rangle = \langle v,u\rangle = \langle v,v\rangle = 0$. □

co-nonn-deg **Corollary 2.4.** *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a nonnegative inner product on $\mathscr{V}$. Then the following two implications are equivalent.*

i-nondeg (i) *If $v \in \mathscr{V}$ and $\langle u, v\rangle = 0$ for all $u \in \mathscr{V}$, then $v = 0$.*

i-pd (ii) *If $v \in \mathscr{V}$ and $\langle v, v\rangle = 0$, then $v = 0$.*

Proof. Assume that the implication (i) holds and let $v \in \mathscr{V}$ be such that $\langle v, v\rangle = 0$. Let $u \in \mathscr{V}$ be arbitrary. By the the CBS inequality

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle = 0.$$

Thus, $\langle u, v\rangle = 0$ for all $u \in \mathscr{V}$. By (i) we conclude $v = 0$. This proves (ii).

The converse is trivial. However, here is a proof. Assume that the implication (ii) holds. To prove (i), let $v \in \mathscr{V}$ and assume $\langle u, v\rangle = 0$ for all $u \in \mathscr{V}$. Setting $u = v$ we get $\langle v, v\rangle = 0$. Now (ii) yields $v = 0$. ☐

Definition 2.5. Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$. An inner product $[\,\cdot\,,\cdot\,]$ on $\mathscr{V}$ is *nondegenerate* if the following implication holds

(d) (nondegenerecy) $\quad u \in \mathscr{V}$ and $[u, v] = 0$ for all $v \in \mathscr{V}$ implies $u = 0$.

We conclude this section with a characterization of the best approximation property.

th-BA-Ort **Theorem 2.6** (Best Approximation-Orthogonality Theorem)**.** *Let $(\mathscr{V}, \langle\cdot,\cdot\rangle)$ be an inner product space with a nonnegative inner product. Let $\mathscr{U}$ be a subspace of $\mathscr{V}$. Let $v \in \mathscr{V}$ and $u_0 \in \mathscr{U}$. Then*

$$\forall\, u \in \mathscr{U} \qquad \langle v - u_0, v - u_0\rangle \leq \langle v - u, v - u\rangle. \tag{9}$$ eq-mind

if and only if

$$\forall\, u \in \mathscr{U} \qquad \langle v - u_0, u\rangle = 0. \tag{10}$$ eq-orth

Proof. First we prove the "only if" part. Assume (9). Let $u \in \mathscr{U}$ be arbitrary. Set $\alpha = \langle v - u_0, u\rangle$. Clearly $\alpha \in \mathbb{F}$. Let $t \in \mathbb{Q} \subseteq \mathbb{F}$ be arbitrary. Since $u_0 - t\alpha u \in \mathscr{U}$, (9) implies

$$\forall\, t \in \mathbb{Q} \qquad \langle v - u_0, v - u_0\rangle \leq \langle v - u_0 + t\alpha u, v - u_0 + t\alpha u\rangle. \tag{11}$$ eq-mind1

Now recall that $\alpha = \langle v - u_0, u\rangle$ and expand the right-hand side of (11):

$$\begin{aligned}\langle v - u_0 + t\alpha u, v - u_0 + t\alpha u\rangle &= \langle v - u_0, v - u_0\rangle + \langle v - u_0, t\alpha u\rangle \\ &\qquad + \langle t\alpha u, v - u_0\rangle + \langle t\alpha u, t\alpha u\rangle \\ &= \langle v - u_0, v - u_0\rangle + t\overline{\alpha}\langle v - u_0, u\rangle\end{aligned}$$

$$\begin{aligned} &+ t\alpha\langle u, v-u_0\rangle + t^2|\alpha|^2\langle u,u\rangle \\ &= \langle v-u_0, v-u_0\rangle + 2t|\alpha|^2 + t^2|\alpha|^2\langle u,u\rangle. \end{aligned}$$

Thus (11) is equivalent to

$$\forall\, t\in\mathbb{Q} \qquad 0 \le 2t|\alpha|^2 + t^2|\alpha|^2\langle u,u\rangle. \tag{12}$$

eq-mind2

By the High School Theorem, (12) implies

$$4|\alpha|^4 - 4|\alpha|^2\langle u,u\rangle\, 0 = 4|\alpha|^4 \le 0.$$

Consequently $\alpha = \langle v-u_0, u\rangle = 0$. Since $u \in \mathscr{U}$ was arbitrary, (10) is proved.

For the "if" part assume that (10) is true. Let $u \in \mathscr{U}$ be arbitrary. Notice that $u_0 - u \in \mathscr{U}$ and calculate

$$\begin{aligned} \langle v-u, v-u\rangle &= \langle v-u_0+u_0-u, v-u_0+u_0-u\rangle \\ \boxed{\text{by (10) and Pythag. thm.}} \quad &= \langle v-u_0, v-u_0\rangle + \langle u_0-u, u_0-u\rangle \\ \boxed{\text{since } \langle u_0-u, u_0-u\rangle \ge 0} \quad &\ge \langle v-u_0, v-u_0\rangle. \end{aligned}$$

This proves (9). □

3 Positive definite inner products

It follows from Corollary 2.4 that a nonnegative inner product $\langle\,\cdot\,,\cdot\,\rangle$ on $\mathscr{V}$ is nondegenerate if and only if $\langle v,v\rangle = 0$ implies $v=0$. A nonnegative nondegenerate inner product is also called *positive definite inner product*. Since positive definite inner products are the most often encountered inner products we give the complete definition as it is commonly given in textbooks.

Definition 3.1. Let $\mathscr{V}$ be a vector space over a scalar field $\mathbb{F}$. A function $\langle\,\cdot\,,\cdot\,\rangle : \mathscr{V}\times\mathscr{V}\to\mathbb{F}$ is called a *positive definite inner product* on $\mathscr{V}$ if the following conditions are satisfied;

(a) $\forall\, u,v,w\in\mathscr{V} \ \ \forall\,\alpha,\beta\in\mathbb{F} \quad \langle \alpha u + \beta v, v\rangle = \alpha\langle u,w\rangle + \beta\langle v,w\rangle$,

(b) $\forall\, u,v\in\mathscr{V} \quad \langle u,v\rangle = \overline{\langle v,u\rangle}$,

(c) $\forall\, v\in\mathscr{V} \quad \langle v,v\rangle \ge 0$,

(d) If $v\in\mathscr{V}$ and $\langle v,v\rangle = 0$, then $v=0$.

A positive definite inner product gives rise to a norm.

Theorem 3.2. *Let $(\mathscr{V}, \langle\cdot,\cdot\rangle)$ be a vector space over $\mathbb{F}$ with a positive definite inner product $\langle\cdot,\cdot\rangle$. The function $\|\cdot\| : \mathscr{V} \to \mathbb{R}$ defined by*

$$\|v\| = \sqrt{\langle v, v\rangle}, \quad v \in \mathscr{V},$$

is a norm on $\mathscr{V}$. That is for all $u, v \in \mathscr{V}$ and all $\alpha \in \mathbb{F}$ we have $\|v\| \geq 0$, $\|\alpha v\| = |\alpha|\|v\|$, $\|u + v\| \leq \|u\| + \|v\|$ and $\|v\| = 0$ implies $v = 0_{\mathscr{V}}$.

Definition 3.3. Let $(\mathscr{V}, \langle\cdot,\cdot\rangle)$ be a vector space over $\mathbb{F}$ with a positive definite inner product $\langle\cdot,\cdot\rangle$. A set of vectors $\mathscr{A} \subset \mathscr{V}$ is said to form an *orthogonal system* in $\mathscr{V}$ if for all $u, v \in \mathscr{A}$ we have $\langle u, v\rangle = 0$ whenever $u \neq v$ and for all $v \in \mathscr{A}$ we have $\langle v, v\rangle > 0$. An orthogonal system $\mathscr{A}$ is called an *orthonormal system* if for all $v \in \mathscr{A}$ we have $\langle v, v\rangle = 1$.

th-os-ec **Theorem 3.4.** *Let $(\mathscr{V}, \langle\cdot,\cdot\rangle)$ be a vector space over $\mathbb{F}$ with a positive definite inner product $\langle\cdot,\cdot\rangle$. Let $n \in \mathbb{N}$ and let $u_1, \ldots, u_n$ be an orthogonal system in $\mathscr{V}$, and set $\mathscr{U} = \operatorname{span}\{u_1, \ldots, u_n\}$. The following statements hold.*

th-os-eci1 (a) *If $u = \sum_{j=1}^{n} \alpha_j u_j$, then for all $j \in \{1, \ldots, n\}$ we have $\alpha_j = \dfrac{\langle u, u_j\rangle}{\langle u_j, u_j\rangle}$. In particular, an orthogonal system is linearly independent.*

th-os-eci2 (b) *For every $v \in \mathscr{V}$ we have*

$$v - \sum_{j=1}^{n} \frac{\langle v, u_j\rangle}{\langle u_j, u_j\rangle} u_j \quad \perp \quad \mathscr{U}.$$

th-os-eci3 (c) *For every $v \in \mathscr{V}$ Bessel's inequality holds*

$$\|v\|^2 \geq \sum_{j=1}^{n} \frac{|\langle v, u_j\rangle|^2}{\langle u_j, u_j\rangle}.$$

The equality holds in Bessel's inequality if and only if $v \in \mathscr{U}$.

Proof. To prove (a), let $u = \sum_{j=1}^{n} \alpha_j u_j$, let $k \in \{1, \ldots, n\}$ be arbitrary, and calculate the inner product with u_k for both sides of the equality. Then, using the linearity of the inner product in the first variable and the fact that $\langle u_j, u_k\rangle = 0$ whenever $j \neq k$ we obtain $\langle u, u_k\rangle = \sum_{j=1}^{n} \alpha_j \langle u_j, u_k\rangle = \alpha_k \langle u_k, u_k\rangle$. Since $\langle u_k, u_k\rangle > 0$, we have $\alpha_k = \dfrac{\langle u, u_k\rangle}{\langle u_k, u_k\rangle}$.

To prove (b) let $v \in \mathscr{V}$ be arbitrary. Let let $k \in \{1, \ldots, n\}$ be arbitrary, and calculate the inner product

$$\left\langle v - \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j, u_k \right\rangle = \langle v, u_k \rangle - \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} \langle u_j, u_k \rangle$$
$$= \langle v, u_k \rangle - \langle v, u_k \rangle$$
$$= 0.$$

Since $k \in \{1, \ldots, n\}$ was arbitrary, replacing u_k with an arbitrary vector in $\mathscr{U}$ also leads to the zero inner product.

To prove (c) we observe that the $n+1$ vectors on the right side in the equality

$$v = \left(v - \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j \right) + \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j$$

are mutually orthogonal and apply the Pythagorean Theorem to obtain

$$\|v\|^2 = \left\| v - \sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j \right\|^2 + \sum_{j=1}^{n} \frac{|\langle v, u_j \rangle|^2}{\langle u_j, u_j \rangle}.$$

Bessel's inequality and the characterization of the equality follow from the preceding equality. □

The formulas that appear in the preceding theorem are probably the most important formulas in positive definite inner product spaces. My nickname for the content in (a) is "easy coefficients" since (a) shows that finding the coefficients of a linear combination of an orthogonal system is given by clear formulas. The vector

$$\sum_{j=1}^{n} \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle} u_j$$

in (b) is called the orthogonal projection of v onto $\mathscr{U}$. For more about the orthogonal projections see paragraphs after Corollary 3.11. My nickname for the content in (b) is "easy orthogonal projection" since (b) shows that finding the coefficients of the orthogonal projection onto a span of an orthogonal system is given by a clear formula. Bessel's inequality needs no nickname, it is one of the key tools in proving convergence of Fourier series.

Theorem 3.5 (The Gram-Schmidt orthogonalization)**.** *Let $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle)$ be a vector space over $\mathbb{F}$ with a positive definite inner product $\langle\,\cdot\,,\cdot\,\rangle$. Let $n \in \mathbb{N}$ and let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathscr{V}$. Let the vectors $u_1, \ldots, u_n$ be defined recursively by*

$$\begin{aligned} u_1 &= v_1, \\ u_{k+1} &= v_{k+1} - \sum_{j=1}^{k} \frac{\langle v_{k+1}, u_j\rangle}{\langle u_j, u_j\rangle} u_j, \quad k \in \{1, \ldots, n-1\}. \end{aligned}$$

Then the vectors $u_1, \ldots, u_n$ form an orthogonal system which has the same fan as the given vectors $v_1, \ldots, v_n$.

Proof. We will prove by Mathematical Induction the following statement: For all $k \in \{1, \ldots, n\}$ we have:

(a) $\langle u_k, u_k\rangle > 0$ and $\langle u_j, u_k\rangle = 0$ whenever $j \in \{1, \ldots, k-1\}$;
(b) vectors $u_1, \ldots, u_k$ are linearly independent;
(c) $\operatorname{span}\{u_1, \ldots, u_k\} = \operatorname{span}\{v_1, \ldots, v_k\}$.

For $k = 1$ statements (a), (b) and (c) are clearly true. Let $m \in \{1, \ldots, n-1\}$ and assume that statements (a), (b) and (c) are true for all $k \in \{1, \ldots, m\}$.

Next we will prove that statements (a), (b) and (c) are true for $k = m+1$. Recall the definition of u_{m+1}:

$$u_{m+1} = v_{m+1} - \sum_{j=1}^{m} \frac{\langle v_{m+1}, u_j\rangle}{\langle u_j, u_j\rangle} u_j.$$

By the Inductive Hypothesis we have $\operatorname{span}\{u_1, \ldots, u_m\} = \operatorname{span}\{v_1, \ldots, v_m\}$. Since $v_1 \ldots, v_{m+1}$ are linearly independent, $v_{m+1} \notin \operatorname{span}\{u_1, \ldots, u_m\}$. Therefore, $u_{m+1} \neq 0_{\mathscr{V}}$. That is $\langle u_{m+1}, u_{m+1}\rangle > 0$. Let $k \in \{1, \ldots, m\}$ be arbitrary. Then by the Inductive Hypothesis we have that $\langle u_j, u_k\rangle = 0$ whenever $j \in \{1, \ldots, m\}$ and $j \neq k$. Therefore,

$$\begin{aligned} \langle u_{m+1}, u_k\rangle &= \langle v_{m+1}, u_k\rangle - \sum_{j=1}^{m} \frac{\langle v_{m+1}, u_j\rangle}{\langle u_j, u_j\rangle} \langle u_j, u_k\rangle \\ &= \langle v_{m+1}, u_k\rangle - \langle v_{m+1}, u_k\rangle \\ &= 0. \end{aligned}$$

This proves claim (a). To prove claim (b) notice that by the Inductive Hypothesis $u_1, \ldots, u_m$ are linearly independent and $u_{m+1} \notin \operatorname{span}\{u_1, \ldots, u_m\}$

since $v_{m+1} \notin \operatorname{span}\{u_1, \dots, u_m\}$. To prove claim (c) notice that the definition of u_{m+1} implies $u_{m+1} \in \operatorname{span}\{v_1, \dots, v_{m+1}\}$. Since by the inductive hypothesis $\operatorname{span}\{u_1, \dots, u_m\} = \operatorname{span}\{v_1, \dots, v_m\}$, we have $\operatorname{span}\{u_1, \dots, u_{m+1}\} \subseteq \operatorname{span}\{v_1, \dots, v_{m+1}\}$. The converse inclusion follows from the fact that $v_{m+1} \in \operatorname{span}\{u_1, \dots, u_{m+1}\}$.

It is clear that the claim of the theorem follows from the claim that has been proven. □

The following two statements are immediate consequences of the Gram-Schmidt orthogonalization process.

Corollary 3.6. *If $\mathscr{V}$ is a finite dimensional vector space with positive definite inner product $\langle \cdot, \cdot \rangle$, then $\mathscr{V}$ has an orthonormal basis.*

c-onb-ut **Corollary 3.7.** *If $\mathscr{V}$ is a complex vector space with positive definite inner product and $T \in \mathscr{L}(\mathscr{V})$ then there exists an orthonormal basis $\mathscr{B}$ such that $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ is upper-triangular.*

Definition 3.8. Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be a positive definite inner product space and $\mathscr{A} \subset \mathscr{V}$. We define $\mathscr{A}^{\perp} = \{v \in \mathscr{V} : \langle v, a \rangle = 0 \ \forall\, a \in \mathscr{A}\}$.

The following is a straightforward proposition.

Proposition 3.9. *Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be a positive definite inner product space and $\mathscr{A} \subset \mathscr{V}$. Then $\mathscr{A}^{\perp}$ is a subspace of $\mathscr{V}$.*

th-fd-ds **Theorem 3.10.** *Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be a positive definite inner product space and let $\mathscr{U}$ be a finite dimensional subspace of $\mathscr{V}$. Then $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^{\perp}$.*

Proof. We first prove that $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^{\perp}$. Note that since $\mathscr{U}$ is a subspace of $\mathscr{V}$, $\mathscr{U}$ inherits the positive definite inner product from $\mathscr{V}$. Thus $\mathscr{U}$ is a finite dimensional positive definite inner product space. Thus there exists an orthonormal basis of $\mathscr{U}$, $\mathscr{B} = \{u_1, u_2, \dots u_k\}$.

Let $v \in \mathscr{V}$ be arbitrary. Then

$$v = \left(\sum_{j=1}^{k} \langle v, u_j \rangle u_j \right) + \left(v - \sum_{j=1}^{k} \langle v, u_j \rangle u_j \right),$$

where the first summand is in $\mathscr{U}$. By Theorem 3.4(b) the second summand is in $\mathscr{U}^{\perp}$. This proves that $\mathscr{V} = \mathscr{U} + \mathscr{U}^{\perp}$.

To prove that the sum is direct, let $w \in \mathscr{U}$ and $w \in \mathscr{U}^{\perp}$. Then $\langle w, w \rangle = 0$. Since $\langle \cdot, \cdot \rangle$ is positive definite, this implies $w = 0_{\mathscr{V}}$. The theorem is proved. □

co-pp **Corollary 3.11.** *Let $(\mathscr{V}, \langle\cdot\,,\cdot\rangle)$ be a positive definite inner product space and let $\mathscr{U}$ be a finite dimensional subspace of $\mathscr{V}$. Then $\left(\mathscr{U}^{\perp}\right)^{\perp} = \mathscr{U}$.*

Recall that an arbitrary direct sum $\mathscr{V} = \mathscr{U} \oplus \mathscr{W}$ gives rise to a projection operator $P_{\mathscr{U}\|\mathscr{W}}$, the projection of $\mathscr{V}$ onto $\mathscr{U}$ parallel to $\mathscr{W}$.

If $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^{\perp}$, then the resulting projection of $\mathscr{V}$ onto $\mathscr{U}$ parallel to $\mathscr{U}^{\perp}$ is called the *orthogonal projection* of $\mathscr{V}$ onto $\mathscr{U}$; it is denoted simply by $P_{\mathscr{U}}$. By definition for every $v \in \mathscr{V}$,

$$u = P_{\mathscr{U}} v \qquad \Leftrightarrow \qquad u \in \mathscr{U} \quad \text{and} \quad v - u \in \mathscr{U}^{\perp}.$$

As for any projection we have $P_{\mathscr{U}} \in \mathscr{L}(\mathscr{V})$, $\operatorname{ran} P_{\mathscr{U}} = \mathscr{U}$, $\operatorname{nul} P_{\mathscr{U}} = \mathscr{U}^{\perp}$, and $(P_{\mathscr{U}})^2 = P_{\mathscr{U}}$.

Theorems 3.10 and 2.6 yield the following solution of the best approximation problem for finite dimensional subspaces of a vector space with a positive definite inner product.

Corollary 3.12. *Let $(\mathscr{V}, \langle\cdot\,,\cdot\rangle)$ be a vector space with a positive definite inner product and let $\mathscr{U}$ be a finite dimensional subspace of $\mathscr{V}$. For arbitrary $v \in \mathscr{V}$ the vector $P_{\mathscr{U}} v \in \mathscr{U}$ is the unique best approximation for v in $\mathscr{U}$. That is*

$$\left\|v - P_{\mathscr{U}} v\right\| < \|v - u\| \quad \text{for all} \quad u \in \mathscr{U} \setminus \left\{P_{\mathscr{U}} v\right\}. \tag{13}$$ eq-bapp

Proof. Let $v \in \mathscr{V}$ and $u \in \mathscr{U} \setminus \left\{P_{\mathscr{U}} v\right\}$ be arbitrary. Recall that the basic facts about the orthogonal projection:

$$P_{\mathscr{U}} v \in \mathscr{U}, \qquad v - P_{\mathscr{U}} v \in \mathscr{U}^{\perp}.$$

In the next calculation we use the preceding two facts, the Pythagorean Theorem and the fact that $u \neq P_{\mathscr{U}} v$ as follows

$$\begin{aligned} \|v - u\|^2 &= \left\|v - P_{\mathscr{U}} v + P_{\mathscr{U}} v - u\right\|^2 \\ &= \left\|v - P_{\mathscr{U}} v\right\|^2 + \left\|P_{\mathscr{U}} v - u\right\|^2 \\ &> \left\|v - P_{\mathscr{U}} v\right\|^2. \end{aligned}$$

Taking the square root of both sides of the preceding inequality proves the corollary. □

4 The definition of an adjoint operator

Let $\mathscr{V}$ be a vector space over $\mathbb{F}$. The space $\mathscr{L}(\mathscr{V}, \mathbb{F})$ is called the *dual space* of $\mathscr{V}$; it is denoted by $\mathscr{V}^*$.

th-Phi **Theorem 4.1.** *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. Define the mapping*

$$\Phi : \mathscr{V} \to \mathscr{V}^*$$

as follows: for $w \in \mathscr{V}$ we set

$$\bigl(\Phi(w)\bigr)(v) = \langle v, w\rangle \quad \text{for all} \quad v \in \mathscr{V}.$$

Then Φ is a anti-linear bijection.

Proof. Clearly, for each $w \in \mathscr{V}$, $\Phi(w) \in \mathscr{V}^*$. The mapping Φ is anti-linear, since for $\alpha, \beta \in \mathbb{F}$ and $u, w \in \mathscr{V}$, for all $v \in \mathscr{V}$ we have

$$\begin{aligned}\bigl(\Phi(\alpha u + \beta w)\bigr)(v) &= \langle v, \alpha u + \beta w\rangle \\ &= \overline{\alpha}\langle v, u\rangle + \overline{\beta}\langle v, w\rangle \\ &= \overline{\alpha}\bigl(\Phi(u)\bigr)(v) + \overline{\beta}\bigl(\Phi(w)\bigr)(v) \\ &= \bigl(\overline{\alpha}\Phi(u) + \overline{\beta}\Phi(w)\bigr)(v).\end{aligned}$$

Thus $\Phi(\alpha u + \beta w) = \overline{\alpha}\Phi(u) + \overline{\beta}\Phi(w)$. This proves anti-linearity.

To prove injectivity of Φ, let $u, w \in \mathscr{V}$ be such that $\Phi(u) = \Phi(w)$. Then $\bigl(\Phi(u)\bigr)(v) = \bigl(\Phi(w)\bigr)(v)$ for all $v \in \mathscr{V}$. By the definition of Φ this means $\langle v, u\rangle = \langle v, w\rangle$ for all $v \in \mathscr{V}$. Consequently, $\langle v, u - w\rangle = 0$ for all $v \in \mathscr{V}$. In particular, with $v = u - w$ we have $\langle u - w, u - w\rangle = 0$. Since $\langle\,\cdot\,,\cdot\,\rangle$ is a positive definite inner product, it follows that $u - w = 0_{\mathscr{V}}$, that is $u = w$.

To prove that Φ is a surjection we use the assumption that $\mathscr{V}$ is finite dimensional. Then there exists an orthonormal basis $u_1, \ldots, u_n$ of $\mathscr{V}$. Let $\varphi \in \mathscr{V}^*$ be arbitrary. Set

$$w = \sum_{j=1}^{n} \overline{\varphi(u_j)}\, u_j.$$

The proof that $\Phi(w) = \varphi$ follows. Let $v \in \mathscr{V}$ be arbitrary.

$$\begin{aligned}\bigl(\Phi(w)\bigr)(v) &= \langle v, w\rangle \\ &= \left\langle v, \sum_{j=1}^{n} \overline{\varphi(u_j)}\, u_j \right\rangle \\ &= \sum_{j=1}^{n} \varphi(u_j)\langle v, u_j\rangle\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \langle v, u_j \rangle \varphi(u_j) \\
&= \varphi\left(\sum_{j=1}^{n} \langle v, u_j \rangle u_j \right) \\
&= \varphi(v).
\end{aligned}
$$

The theorem is proved. □

pr-alb **Proposition 4.2.** *Let $\mathscr{V}$ and $\mathscr{W}$ be vector spaces over $\mathbb{F}$ and let $\mathscr{V}$ be finite-dimensional. If $\Psi : \mathscr{V} \to \mathscr{W}$ is an anti-linear bijection, then $\mathscr{W}$ is finite-dimensional and* $\dim \mathscr{V} = \dim \mathscr{W}$.

Proof. Let $n = \dim \mathscr{V}$ and let $u_1, \ldots, u_n$ be a basis for $\mathscr{V}$. We will prove that $\Psi(u_1), \ldots, \Psi(u_n)$ is a basis for $\mathscr{W}$. First we prove that $\Psi(u_1), \ldots, \Psi(u_n)$ are linearly independent. For this goal, let $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ be such that

$$\alpha_1 \Psi(u_1) + \cdots + \alpha_n \Psi(u_n) = 0_{\mathscr{W}}.$$

Since $\Psi : \mathscr{V} \to \mathscr{W}$ is anti-linear, the last equality is equivalent to

$$\Psi\big(\overline{\alpha_1} u_1 + \cdots + \overline{\alpha_n} u_n\big) = 0_{\mathscr{W}}.$$

Consequently, since Ψ is anti-linear bijection, we have

$$\overline{\alpha_1} u_1 + \cdots + \overline{\alpha_n} u_n = 0_{\mathscr{V}}.$$

Since $u_1, \ldots, u_n$ are linearly independent, we deduce that for all $k \in \{1, \ldots, n\}$ we have $\overline{\alpha_k} = 0_{\mathbb{F}}$. Therefore for all $k \in \{1, \ldots, n\}$ we have $\alpha_k = \overline{\overline{\alpha_k}} = \overline{0_{\mathbb{F}}} = 0_{\mathbb{F}}$. This proves linear independence.

Now we prove that $\Psi(u_1), \ldots, \Psi(u_n)$ span $\mathscr{W}$. Let $w \in \mathscr{W}$ be arbitrary. Since $\Psi : \mathscr{V} \to \mathscr{W}$ is a surjection there exists $v \in \mathscr{V}$ such that $\Psi(v) = w$. Since the vectors $u_1, \ldots, u_n$ span $\mathscr{V}$, there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that

$$v = \alpha_1 u_1 + \cdot + \alpha_n u_n.$$

Applying Ψ to both sides of the preceding equality and using that Ψ is anti-linear, we obtain

$$w = \Psi(v) = \Psi\big(\alpha_1 u_1 + \cdot + \alpha_n u_n\big) = \overline{\alpha_1} \Psi(u_1) + \cdots + \overline{\alpha_n} \Psi(u_n).$$

Thus, w is a linear combination of $\Psi(u_1), \ldots, \Psi(u_n)$. Since $w \in \mathscr{W}$ was arbitrary, the vectors $\Psi(u_1), \ldots, \Psi(u_n)$ span $\mathscr{W}$. This proves that $\Psi(u_1), \ldots, \Psi(u_n)$ is a basis for $\mathscr{W}$. Thus $\dim \mathscr{V} = \dim \mathscr{W}$. □

Corollary 4.3. *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. Then* $\dim \mathscr{V} = \dim \mathscr{V}^*$.

Proof. Since $\Phi : \mathscr{V} \to \mathscr{V}^*$ from Theorem 4.1 is an anti-linear bijection, Proposition 4.2 implies that $\dim \mathscr{V} = \dim \mathscr{V}*$. $\square$

The mapping Φ from Theorem 4.1 is a convenient tool for defining the adjoint of a linear operator. In the following definition, we will deal with two positive definite inner product spaces $(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}})$ and $(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}})$. We will use subscripts to emphasize the inner products and different mappings Φ:

$$\Phi_{\mathscr{V}} : \mathscr{V} \to \mathscr{V}^*, \qquad \Phi_{\mathscr{W}} : \mathscr{W} \to \mathscr{W}^*.$$

Recall that for every $x, v \in \mathscr{V}$ we have

$$\big(\Phi_{\mathscr{V}}(v)\big)(x) = \langle x, v\rangle_{\mathscr{V}}$$

and for every $y, w \in \mathscr{W}$ we have

$$\big(\Phi_{\mathscr{W}}(w)\big)(y) = \langle y, w\rangle_{\mathscr{W}}.$$

Let $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$ be two finite dimensional vector spaces over the same scalar field $\mathbb{F}$ and with positive definite inner products. Let $T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$. We define the adjoint $T^* : \mathscr{W} \to \mathscr{V}$ of T by

$$T^* w = \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w) \circ T\big), \qquad w \in \mathscr{W}. \tag{14}$$

eq-def-T*

Since $\Phi_{\mathscr{W}}$ and $\Phi_{\mathscr{V}}^{-1}$ are anti-linear, T^* is linear. For arbitrary $\alpha_1, \alpha_1 \in \mathbb{F}$ and $w_1, w_2 \in \mathscr{W}$ we have

$$\begin{aligned}
T^*(\alpha_1 w_1 + \alpha_2 w_2) &= \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(\alpha_1 w_1 + \alpha_2 w_2) \circ T\big) \\
&= \Phi_{\mathscr{V}}^{-1}\big(\big(\overline{\alpha}_1 \Phi_{\mathscr{W}}(w_1) + \overline{\alpha}_2 \Phi_{\mathscr{W}}(w_2)\big) \circ T\big) \\
&= \Phi_{\mathscr{V}}^{-1}\big(\overline{\alpha}_1\, \Phi_{\mathscr{W}}(w_1) \circ T + \overline{\alpha}_2\, \Phi_{\mathscr{W}}(w_2) \circ T\big) \\
&= \alpha_1\, \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w_1) \circ T\big) + \alpha_2\, \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w_2) \circ T\big) \\
&= \alpha_1\, T^* w_1 + \alpha_2\, T^* w_2.
\end{aligned}$$

Thus, $T^* \in \mathscr{L}(\mathscr{W}, \mathscr{V})$.

Next we will deduce the most important property of T^*. By the definition of $T^* : \mathscr{W} \to \mathscr{V}$, for a fixed arbitrary $w \in \mathscr{W}$ we have

$$T^* w = \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w) \circ T\big).$$

This is equivalent to

$$\Phi_{\mathscr{V}}\big(T^*w\big) = \Phi_{\mathscr{W}}(w) \circ T,$$

which is, by the definition of $\Phi_{\mathscr{V}}$, equivalent to

$$\big(\Phi_{\mathscr{W}}(w) \circ T\big)(v) = \langle v, T^*w\rangle_{\mathscr{V}} \quad \text{for all} \quad v \in \mathscr{V},$$

which, in turn, is equivalent to

$$\big(\Phi_{\mathscr{W}}(w)\big)(Tv) = \langle v, T^*w\rangle_{\mathscr{V}} \quad \text{for all} \quad v \in \mathscr{V}.$$

From the definition of $\Phi_{\mathscr{W}}$ the last statement is equivalent to

$$\langle Tv, w\rangle_{\mathscr{W}} = \langle v, T^*w\rangle_{\mathscr{V}} \quad \text{for all} \quad v \in \mathscr{V}.$$

The reasoning above proves the following proposition.

p-ch-adj **Proposition 4.4.** *Let $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$ be two finite dimensional vector spaces over the same scalar field $\mathbb{F}$ and with positive definite inner products. Let $T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$ and $S \in \mathscr{L}(\mathscr{W}, \mathscr{V})$. Then $S = T^*$ if and only if*

$$\langle Tv, w\rangle_{\mathscr{W}} = \langle v, Sw\rangle_{\mathscr{V}} \quad \textit{for all} \quad v \in \mathscr{V}, w \in \mathscr{W}. \tag{15}$$ eq-def-T*e

5 Properties of the adjoint operator

Theorem 5.1. *Let $\big(\mathscr{U}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{U}}\big)$, $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$ be three finite dimensional vector space over the same scalar field $\mathbb{F}$ and with positive definite inner products. Let $S \in \mathscr{L}(\mathscr{U}, \mathscr{V})$ and $T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$. Then $(TS)^* = S^*T^*$.*

Proof. By definition for every $u \in \mathscr{U}$, $v \in \mathscr{V}$ and $w \in \mathscr{W}$ we have

$$\begin{aligned} S^*v &= \Phi_{\mathscr{U}}^{-1}\big(\Phi_{\mathscr{V}}(v) \circ S\big) \\ T^*w &= \Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w) \circ T\big) \\ (TS)^*w &= \Phi_{\mathscr{U}}^{-1}\big(\Phi_{\mathscr{W}}(w) \circ (TS)\big) \end{aligned}$$

With this, for arbitrary $w \in \mathscr{W}$ we calculate

$$\begin{aligned} S^*T^*w &= S^*(T^*w) \\ &= \Phi_{\mathscr{U}}^{-1}\Big(\Phi_{\mathscr{V}}\big(\Phi_{\mathscr{V}}^{-1}\big(\Phi_{\mathscr{W}}(w) \circ T\big)\big) \circ S\Big) \\ &= \Phi_{\mathscr{U}}^{-1}\Big(\Phi_{\mathscr{W}}(w) \circ T \circ S\Big) \\ &= (TS)^*w. \end{aligned}$$

Thus $(TS)^* = S^*T^*$. $\square$

A function $f : X \to X$ is said to be an *involution* if it is its own inverse, that is if $f(f(x)) = x$ for all $x \in X$.

th-pr-adj-in **Theorem 5.2.** *Let $\bigl(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\bigr)$ and $\bigl(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\bigr)$ be two finite dimensional vector spaces over the same scalar field $\mathbb{F}$ and with positive definite inner products. The adjoint mapping*

$$^{*} : \mathscr{L}(\mathscr{V}, \mathscr{W}) \to \mathscr{L}(\mathscr{W}, \mathscr{V})$$

is an anti-linear bijection. Its inverse is the adjoint mapping from $\mathscr{L}(\mathscr{W}, \mathscr{V})$ to $\mathscr{L}(\mathscr{V}, \mathscr{W})$. In particular the adjoint mapping in $\mathscr{L}(\mathscr{V}, \mathscr{V})$ is an anti-linear involution.

Proof. To prove that $\ ^{*} : \mathscr{L}(\mathscr{V}, \mathscr{W}) \to \mathscr{L}(\mathscr{W}, \mathscr{V})$ is anti-linear let $\alpha, \beta \in \mathbb{F}$ be arbitrary and let $S, T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$ be arbitrary. By the definition of $\ ^{*}$ for arbitrary $w \in \mathscr{W}$ we have

$$\begin{aligned}
(\alpha S + \beta T)^{*} w &= \Phi_{\mathscr{V}}^{-1}\bigl(\Phi_{\mathscr{W}}(w) \circ (\alpha S + \beta T)\bigr) \\
&= \Phi_{\mathscr{V}}^{-1}\bigl(\alpha \Phi_{\mathscr{W}}(w) \circ S + \beta \Phi_{\mathscr{W}}(w) \circ T\bigr) \\
&= \overline{\alpha} \Phi_{\mathscr{V}}^{-1}\bigl(\Phi_{\mathscr{W}}(w) \circ S\bigr) + \overline{\beta} \Phi_{\mathscr{V}}^{-1}\bigl(\Phi_{\mathscr{W}}(w) \circ T\bigr) \\
&= \overline{\alpha} S^{*} w + \overline{\beta} T^{*} w \\
&= \bigl(\overline{\alpha} S^{*} + \overline{\beta} T^{*}\bigr) w.
\end{aligned}$$

Hence $(\alpha S + \beta T)^{*} = \overline{\alpha} S^{*} + \overline{\beta} T^{*}$.

To prove that the adjoint mapping $\ ^{*} : \mathscr{L}(\mathscr{V}, \mathscr{W}) \to \mathscr{L}(\mathscr{W}, \mathscr{V})$ is a bijection we will use the adjoint mapping $\ ^{\star} : \mathscr{L}(\mathscr{W}, \mathscr{V}) \to \mathscr{L}(\mathscr{V}, \mathscr{W})$. In fact we will prove that $\ ^{\star}$ is the inverse of $\ ^{*}$. To this end we will prove that for all $S \in \mathscr{L}(\mathscr{V}, \mathscr{W})$ we have that $(S^{*})^{\star} = S$ and that for all $T \in \mathscr{L}(\mathscr{W}, \mathscr{V})$ we have that $(T^{\star})^{*} = T$.

Here are the proofs. By the definition of the mapping $\ ^{*} : \mathscr{L}(\mathscr{V}, \mathscr{W}) \to \mathscr{L}(\mathscr{W}, \mathscr{V})$ for an arbitrary $S \in \mathscr{L}(\mathscr{V}, \mathscr{W})$ we have

$$\forall v \in \mathscr{V} \quad \forall w \in \mathscr{W} \qquad \langle S^{*} w, v \rangle_{\mathscr{V}} = \langle w, S v \rangle_{\mathscr{W}}.$$

By Proposition 4.4 this identity yields $(S^{*})^{\star} = S$. By the definition of the mapping $\ ^{\star} : \mathscr{L}(\mathscr{W}, \mathscr{V}) \to \mathscr{L}(\mathscr{V}, \mathscr{W})$ for an arbitrary $T \in \mathscr{L}(\mathscr{W}, \mathscr{V})$ we have

$$\forall w \in \mathscr{W} \quad \forall v \in \mathscr{V} \qquad \langle T^{\star} v, w \rangle_{\mathscr{W}} = \langle v, T w \rangle_{\mathscr{V}}.$$

By Proposition 4.4 this identity yields $(T^{\star})^{*} = T$. ☐

th-pr-adj **Theorem 5.3.** *Let $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$ be two finite dimensional vector spaces over the same scalar field $\mathbb{F}$ and with positive definite inner products. The following statements hold.*

(i) $\operatorname{nul}(T^*) = (\operatorname{ran} T)^{\perp}$.
(ii) $\operatorname{ran}(T^*) = (\operatorname{nul} T)^{\perp}$.
(iii) $\operatorname{nul}(T) = (\operatorname{ran} T^*)^{\perp}$.
(iv) $\operatorname{ran}(T) = (\operatorname{nul} T^*)^{\perp}$.

th-adj-mat **Theorem 5.4.** *Let $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$ be two finite dimensional vector spaces over the same scalar field $\mathbb{F}$ and with positive definite inner products. Let $\mathscr{B}$ and $\mathscr{C}$ be orthonormal bases of $\big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$ and $\big(\mathscr{W}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{W}}\big)$, respectively, and let $T \in \big(\mathscr{V}, \langle\,\cdot\,,\cdot\,\rangle_{\mathscr{V}}\big)$. Then $\mathsf{M}_{\mathscr{B}}^{\mathscr{C}}(T^*)$ is the conjugate transpose of the matrix $\mathsf{M}_{\mathscr{C}}^{\mathscr{B}}(T)$.*

Proof. Let $\mathscr{B} = \{v_1, \ldots, v_m\}$ and $\mathscr{C} = \{w_1, \ldots, w_n\}$ be orthonormal bases from the theorem. Let $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Then the term in the j-th column and the i-th row of the $n \times m$ matrix $\mathsf{M}_{\mathscr{C}}^{\mathscr{B}}(T)$ is $\langle T v_j, w_i \rangle$, while the term in the i-th column and the j-th row of the $m \times n$ matrix $\mathsf{M}_{\mathscr{B}}^{\mathscr{C}}(T^*)$ is

$$\langle T^* w_i, v_j \rangle = \langle w_i, T v_j \rangle = \overline{\langle T v_j, w_i \rangle}.$$

This proves the claim. □

le-Uinv **Lemma 5.5.** *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. Let $\mathscr{U}$ be a subspace of $\mathscr{V}$ and let $T \in \mathscr{L}(\mathscr{V})$. The subspace $\mathscr{U}$ is invariant under T if and only if the subspace $\mathscr{U}^{\perp}$ is invariant under T^*.*

Proof. By the definition of adjoint we have

$$\langle T u, v \rangle = \langle u, T^* v \rangle \tag{16}$$ eq-ad-1

for all $u, v \in \mathscr{V}$. Assume $T\mathscr{U} \subseteq \mathscr{U}$. From (16) we get

$$0 = \langle T u, v \rangle = \langle u, T^* v \rangle \qquad \forall\, u \in \mathscr{U} \quad \text{and} \quad \forall v \in \mathscr{U}^{\perp}.$$

Therefore, $T^* v \in \mathscr{U}^{\perp}$ for all $v \in \mathscr{U}^{\perp}$. This proves “only if” part.

The proof of the “if” part is similar. □

6 Self-adjoint and normal operators

Definition 6.1. Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. An operator $T \in \mathscr{L}(\mathscr{V})$ is said to be *self-adjoint* if $T = T^*$. An operator $T \in \mathscr{L}(\mathscr{V})$ is said to be *normal* if $TT^* = T^*T$.

Proposition 6.2. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. All eigenvalues of a self-adjoint $T \in \mathscr{L}(\mathscr{V})$ are real.*

Proof. Let $\lambda \in \mathbb{F}$ be an eigenvalue of T and let $Tv = \lambda v$ with a nonzero $v \in \mathscr{V}$. Then

$$\lambda\langle v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle.$$

Since $\langle v, v\rangle > 0$ the preceding equalities yield $\lambda = \overline{\lambda}$. $\square$

> In the rest of this section we will consider only scalar fields $\mathbb{F}$ which contain the imaginary unit i.

Proposition 6.3. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. Let $T \in \mathscr{L}(\mathscr{V})$. Then $T = 0$ if and only if $\langle Tv, v\rangle = 0$ for all $v \in \mathscr{V}$.*

Proof. Set, $[u, v] = \langle Tu, v\rangle$ for all $u, v \in \mathscr{V}$. Then $[\,\cdot\,,\cdot\,]$ is a sesquilinear form on $\mathscr{V}$. Since $\langle\,\cdot\,,\cdot\,\rangle$ is a positive definite inner product, $T = 0$ if and only if for all $u, v \in \mathscr{V}$ we have $\langle Tu, v\rangle = 0$, which in turn is equivalent to for all $u, v \in \mathscr{V}$ we have $[u, v] = 0$. By Corollary 1.5 $[u, v] = 0$ for all $u, v \in \mathscr{V}$ is equivalent to $[u, u] = 0$ for all $u \in \mathscr{V}$, that is to $\langle Tu, u\rangle = 0$ for all $u \in \mathscr{V}$. $\square$

Proposition 6.4. *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. An operator $T \in \mathscr{L}(\mathscr{V})$ is self-adjoint if and only if $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in \mathscr{V}$.*

Proof. $\square$

`th-no-iff` **Theorem 6.5.** *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$ and let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$. An operator $T \in \mathscr{L}(\mathscr{V})$ is normal if and only if $\|Tv\| = \|T^*v\|$ for all $v \in \mathscr{V}$.*

`co-noT-sym-Sp` **Corollary 6.6.** *Let $\mathscr{V}$ be a vector space over $\mathbb{F}$, let $\langle\,\cdot\,,\cdot\,\rangle$ be a positive definite inner product on $\mathscr{V}$ and let $T \in \mathscr{L}(\mathscr{V})$ be normal. Then $\lambda \in \mathbb{C}$ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^* and*

$$\operatorname{nul}\big(T^* - \overline{\lambda}I\big) = \operatorname{nul}(T - \lambda I).$$

7 The Spectral Theorem

> In the rest of the notes we will consider only the scalar field $\mathbb{C}$.

th-charnor **Theorem 7.1.** *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ and $\langle \cdot, \cdot \rangle$ be a positive definite inner product on $\mathscr{V}$. Let $T \in \mathscr{L}(\mathscr{V})$. Then $\mathscr{V}$ has an orthonormal basis which consists of eigenvectors of T if and only if T is normal. In other words, T is normal if and only if there exists an orthonormal basis $\mathscr{B}$ of $\mathscr{V}$ such that $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ is a diagonal matrix.*

Proof. Let $n = \dim(\mathscr{V})$. Assume that T is normal. By Corollary 3.7 there exists an orthonormal basis $\mathscr{B} = \{u_1, \ldots, u_n\}$ of $\mathscr{V}$ such that $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ is upper-triangular. That is,

$$\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T) = \begin{bmatrix} \langle Tu_1, u_1 \rangle & \langle Tu_2, u_1 \rangle & \cdots & \langle Tu_n, u_1 \rangle \\ 0 & \langle Tu_2, u_2 \rangle & \cdots & \langle Tu_n, u_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle Tu_n, u_n \rangle \end{bmatrix}, \tag{17}$$ eq-MBBut

or, equivalently,

$$Tu_k = \sum_{j=1}^{k} \langle Tu_k, u_j \rangle u_j \quad \text{for all} \quad k \in \{1, \ldots, n\}. \tag{18}$$ eq-Tuk

By Theorem 5.4 we have

$$\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T^*) = \begin{bmatrix} \overline{\langle Tu_1, u_1 \rangle} & 0 & \cdots & 0 \\ \overline{\langle Tu_2, u_1 \rangle} & \overline{\langle Tu_2, u_2 \rangle} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\langle Tu_n, u_1 \rangle} & \overline{\langle Tu_n, u_2 \rangle} & \cdots & \overline{\langle Tu_n, u_n \rangle} \end{bmatrix}.$$

Consequently,

$$T^* u_k = \sum_{j=k}^{n} \overline{\langle Tu_j, u_k \rangle} u_j \quad \text{for all} \quad k \in \{1, \ldots, n\}. \tag{19}$$ eq-T*uk

Since T is normal, Theorem 6.5 implies

$$\|Tu_k\|^2 = \|T^* u_k\|^2 \quad \text{for all} \quad k \in \{1, \ldots, n\}.$$

Together with (18) and (19) the last identities become

$$\sum_{j=1}^{k} \big|\langle Tu_k, u_j\rangle\big|^2 = \sum_{j=k}^{n} \big|\overline{\langle Tu_j, u_k\rangle}\big|^2 \quad \text{for all} \quad k \in \{1,\dots,n\},$$

or, equivalently,

$$\sum_{j=1}^{k} \big|\langle Tu_k, u_j\rangle\big|^2 = \sum_{j=k}^{n} \big|\langle Tu_j, u_k\rangle\big|^2 \quad \text{for all} \quad k \in \{1,\dots,n\}. \tag{20}$$

eq-sums-eq

The equality in (20) corresponding to $k=1$ reads

$$\big|\langle Tu_1, u_1\rangle\big|^2 = \big|\langle Tu_1, u_1\rangle\big|^2 + \sum_{j=2}^{n} \big|\langle Tu_j, u_1\rangle\big|^2,$$

which implies

$$\langle Tu_j, u_1\rangle = 0 \quad \text{for all} \quad j \in \{2,\dots,n\} \tag{21}$$

eq-1st-row

In other words we have proved that the off-diagonal entries in the first row of the upper triangular matrix $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ in (17) are all zero.

Substituting the value $\langle Tu_2, u_1\rangle = 0$ (from (21)) in the equality in (20) corresponding to $k=2$ reads we get

$$\big|\langle Tu_2, u_2\rangle\big|^2 = \big|\langle Tu_2, u_2\rangle\big|^2 + \sum_{j=3}^{n} \big|\langle Tu_j, u_2\rangle\big|^2,$$

which implies

$$\langle Tu_j, u_2\rangle = 0 \quad \text{for all} \quad j \in \{3,\dots,n\} \tag{22}$$

eq-2nd-row

In other words we have proved that the off-diagonal entries in the second row of the upper triangular matrix $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ in (17) are all zero.

Repeating this reasoning $n-2$ more times would prove that all the off-diagonal entries of the upper triangular matrix $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ in (17) are zero. That is, $\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T)$ is a diagonal matrix.

To prove the converse, assume that there exists an orthonormal basis $\mathscr{B} = \{u_1,\dots,u_n\}$ of $\mathscr{V}$ which consists of eigenvectors of T. That is, for some $\lambda_j \in \mathbb{C}$,

$$Tu_j = \lambda_j u_j \quad \text{for all} \quad j \in \{1,\dots,n\},$$

Then, for arbitrary $v \in \mathscr{V}$ we have

$$Tv = T\left(\sum_{j=1}^{n} \langle v, u_j\rangle u_j\right) = \sum_{j=1}^{n} \langle v, u_j\rangle Tu_j = \sum_{j=1}^{n} \lambda_j \langle v, u_j\rangle u_j. \tag{23}$$

Therefore, for arbitrary $k \in \{1, \dots, n\}$ we have

$$\langle Tv, u_k \rangle = \lambda_k \langle v, u_k \rangle. \tag{24}$$

Now we calculate

$$\begin{aligned}
T^*Tv &= \sum_{j=1}^{n} \langle T^*Tv, u_j \rangle u_j \\
&= \sum_{j=1}^{n} \langle Tv, Tu_j \rangle u_j \\
&= \sum_{j=1}^{n} \langle Tv, Tu_j \rangle u_j \\
&= \sum_{j=1}^{n} \overline{\lambda}_j \langle Tv, u_j \rangle u_j \\
&= \sum_{j=1}^{n} \lambda_j \overline{\lambda}_j \langle v, u_j \rangle u_j.
\end{aligned}$$

Similarly,

$$\begin{aligned}
TT^*v &= T\left(\sum_{j=1}^{n} \langle T^*v, u_j \rangle u_j \right) \\
&= \sum_{j=1}^{n} \langle v, Tu_j \rangle Tu_j \\
&= \sum_{j=1}^{n} \langle v, \lambda_j u_j \rangle \lambda_j u_j \\
&= \sum_{j=1}^{n} \lambda_j \overline{\lambda}_j \langle v, u_j \rangle u_j.
\end{aligned}$$

Thus, we proved $T^*Tv = TT^*v$, that is, T is normal. □

A different proof of the "only if" part of the spectral theorem for normal operators follows. In this proof we use δ_{ij} to represent the Kronecker delta function; that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

Proof. Set $n = \dim \mathscr{V}$. We first prove "only if" part. Assume that T is normal. Set

$$\mathbb{K} = \left\{ k \in \{1, \ldots, n\} \ : \ \begin{array}{l} \exists\, w_1, \ldots, w_k \in \mathscr{V} \quad \text{and} \quad \exists\, \lambda_1, \ldots, \lambda_k \in \mathbb{C} \\ \text{such that } \langle w_i, w_j \rangle = \delta_{ij} \text{ and } Tw_j = \lambda_j w_j \\ \text{for all} \quad i, j \in \{1, \ldots, k\} \end{array} \right\}$$

Clearly $1 \in \mathbb{K}$. Since $\mathbb{K}$ is finite, $m = \max \mathbb{K}$ exists. Clearly, $m \leq n$.

Next we will prove that $k \in \mathbb{K}$ and $k < n$ implies that $k + 1 \in \mathbb{K}$. Assume $k \in \mathbb{K}$ and $k < n$. Let $w_1, \ldots, w_k \in \mathscr{V}$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ be such that $\langle w_i, w_j \rangle = \delta_{ij}$ and $Tw_j = \lambda_j w_j$ for all $i, j \in \{1, \ldots, k\}$. Set

$$\mathscr{W} = \operatorname{span}\{w_1, \ldots, w_k\}.$$

Since $w_1, \ldots, w_k$ are eigenvectors of T we have $T\mathscr{W} \subseteq \mathscr{W}$. By Lemma 5.5, $T^*\big(\mathscr{W}^\perp\big) \subseteq \mathscr{W}^\perp$. Thus, $T^*|_{\mathscr{W}^\perp} \in \mathscr{L}\big(\mathscr{W}^\perp\big)$. Since $\dim \mathscr{W} = k < n$ we have $\dim\big(\mathscr{W}^\perp\big) = n - k \geq 1$. Since $\mathscr{W}^\perp$ is a complex vector space the operator $T^*|_{\mathscr{W}^\perp}$ has an eigenvalue μ with the corresponding unit eigenvector u. Clearly, $u \in \mathscr{W}^\perp$ and $T^*u = \mu u$. Since T^* is normal, Corollary 6.6 yields that $Tu = \overline{\mu} u$. Since $u \in \mathscr{W}^\perp$ and $Tu = \overline{\mu} u$, setting $w_{k+1} = u$ and $\lambda_{k+1} = \overline{\mu}$ we have

$$\langle w_i, w_j \rangle = \delta_{ij} \quad \text{and} \quad Tw_j = \lambda_j w_j \quad \text{for all} \quad i, j \in \{1, \ldots, k, k+1\}.$$

Thus $k + 1 \in \mathbb{K}$. Consequently, $k < m$. Thus, for $k \in \mathbb{K}$, we have proved the implication

$$k < n \qquad \Rightarrow \qquad k < m.$$

The contrapositive of this implication is: For $k \in \mathbb{K}$, we have

$$k \geq m \qquad \Rightarrow \qquad k \geq n.$$

In particular, for $m \in \mathbb{K}$ we have $m = m$ implies $m \geq n$. Since $m \leq n$ is also true, this proves that $m = n$. That is, $n \in \mathbb{K}$. This implies that there exist $u_1, \ldots, u_n \in \mathscr{V}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that $\langle u_i, u_j \rangle = \delta_{ij}$ and $Tu_j = \lambda_j u_j$ for all $i, j \in \{1, \ldots, n\}$.

Since $u_1, \ldots, u_n$ are orthonormal, they are linearly independent. Since $n = \dim \mathscr{V}$, it turns out that $u_1, \ldots, u_n$ form a basis of $\mathscr{V}$. This completes the proof. □

8 Invariance under a normal operator

th-normo-inv **Theorem 8.1.** *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$. Let $\langle\cdot,\cdot\rangle$ be a positive definite inner product on $\mathscr{V}$. Let $T \in \mathscr{L}(\mathscr{V})$ be normal and let $\mathscr{U}$ be a subspace of $\mathscr{V}$. Then*

$$T\mathscr{U} \subseteq \mathscr{U} \quad \Leftrightarrow \quad T\mathscr{U}^\perp \subseteq \mathscr{U}^\perp$$

(Recall that we have previously proved that for any $T \in \mathscr{L}(\mathscr{V})$, $T\mathscr{U} \subseteq \mathscr{U} \Leftrightarrow T^*\mathscr{U}^\perp \subseteq \mathscr{U}^\perp$. Hence if T is normal, showing that any one of $\mathscr{U}$ or $\mathscr{U}^\perp$ is invariant under either T or T^* implies that the rest are, also.)

Proof. Assume $T\mathscr{U} \subseteq \mathscr{U}$. We know $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^\perp$. Let $u_1, \ldots, u_m$ be an orthonormal basis of $\mathscr{U}$ and $u_{m+1}, \ldots, u_n$ be an orthonormal basis of $\mathscr{U}^\perp$. Then $u_1, \ldots, u_n$ is an orthonormal basis of $\mathscr{V}$. If $j \in \{1, \ldots, m\}$ then $u_j \in \mathscr{U}$, so $Tu_j \in \mathscr{U}$. Hence

$$Tu_j = \sum_{k=1}^{m} \langle Tu_j, u_k\rangle u_k.$$

Also, clearly,

$$T^*u_j = \sum_{k=1}^{n} \langle T^*u_j, u_k\rangle u_k.$$

By normality of T we have $\|Tu_j\|^2 = \|T^*u_j\|^2$ for all $j \in \{1, \ldots, m\}$. Starting with this, we calculate

$$\sum_{j=1}^{m} \|Tu_j\|^2 = \sum_{j=1}^{m} \|T^*u_j\|^2$$

$$\boxed{\text{Pythag. thm.}} = \sum_{j=1}^{m}\sum_{k=1}^{n} |\langle T^*u_j, u_k\rangle|^2$$

$$\boxed{\text{group terms}} = \sum_{j=1}^{m}\sum_{k=1}^{m} |\langle T^*u_j, u_k\rangle|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2$$

$$\boxed{\text{def. of } T^*} = \sum_{j=1}^{m}\sum_{k=1}^{m} |\langle u_j, Tu_k\rangle|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2$$

$$\boxed{|\alpha| = |\overline{\alpha}|} = \sum_{j=1}^{m}\sum_{k=1}^{m} |\langle Tu_k, u_j\rangle|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2$$

$$\boxed{\text{order of sum.}} = \sum_{k=1}^{m}\sum_{j=1}^{m} |\langle Tu_k, u_j\rangle|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2$$

$$\boxed{\text{Pythag. thm.}} = \sum_{k=1}^{m} \|Tu_k\|^2 + \sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2.$$

From the above equality we deduce that $\sum_{j=1}^{m}\sum_{k=m+1}^{n} |\langle T^*u_j, u_k\rangle|^2 = 0$. As each term is nonnegative, we conclude that $|\langle T^*u_j, u_k\rangle|^2 = |\langle u_j, Tu_k\rangle|^2 = 0$, that is,

$$\langle u_j, Tu_k\rangle = 0 \quad \text{for all} \quad j \in \{1,\ldots,m\},\ k \in \{m+1,\ldots,n\}. \tag{25}$$ eq-T*-bv

Let now $w \in \mathscr{U}^\perp$ be arbitrary. Then

$$\begin{aligned} Tw &= \sum_{j=1}^{n} \langle Tw, u_j\rangle u_j \\ &= \sum_{j=1}^{n} \Big\langle \sum_{k=m+1}^{n} \langle w, u_k\rangle Tu_k, u_j \Big\rangle u_j \\ &= \sum_{j=1}^{n}\sum_{k=m+1}^{n} \langle w, u_k\rangle \langle Tu_k, u_j\rangle u_j \\ \boxed{\text{by (25)}} &= \sum_{j=m+1}^{n}\sum_{k=m+1}^{n} \langle w, u_k\rangle \langle Tu_k, u_j\rangle u_j \end{aligned}$$

Hence $Tw \in \mathscr{U}^\perp$, that is $T\mathscr{U}^\perp \subseteq \mathscr{U}^\perp$. ☐

A different proof follows. The proof below uses the property of polynomials that for arbitrary distinct $\alpha_1,\ldots,\alpha_m \in \mathbb{C}$ and arbitrary $\beta_1,\ldots,\beta_m \in \mathbb{C}$ there exists a polynomial $p(z) \in \mathbb{C}[z]_{<m}$ such that $p(\alpha_j) = \beta_j$, $j \in \{1,\ldots,m\}$.

Proof. Assume T is normal. By Theorem 7.1 there exists an orthonormal basis $\{u_1,\ldots,u_n\}$ and $\{\lambda_1,\ldots,\lambda_n\} \subseteq \mathbb{C}$ such that

$$Tu_j = \lambda_j u_j \quad \text{for all} \quad j \in \{1,\ldots,n\}.$$

Consequently,

$$T^*u_j = \overline{\lambda_j} u_j \quad \text{for all} \quad j \in \{1,\ldots,n\}.$$

Let v be arbitrary in $\mathscr{V}$. Applying T and T^* to the expansion of v in the basis vectors $\{u_1, \dots, u_n\}$ we obtain

$$Tv = \sum_{j=1}^{n} \lambda_j \langle v, u_j \rangle u_j$$

and

$$T^*v = \sum_{j=1}^{n} \overline{\lambda_j} \langle v, u_j \rangle u_j.$$

Let $p(z) = a_0 + a_1 z + \cdots + a_m z^m \in \mathbb{C}[z]$ be such that

$$p(\lambda_j) = \overline{\lambda}_j \quad \text{for all} \quad j \in \{1, \dots, n\}.$$

Clearly, for all $j \in \{1, \dots, n\}$ we have

$$p(T)u_j = p(\lambda_j)u_j = \overline{\lambda}_j u_j = T^* u_j.$$

Therefore $p(T) = T^*$.

Now assume $T\mathscr{U} \subseteq \mathscr{U}$. Then $T^k\mathscr{U} \subseteq \mathscr{U}$ for all $k \in \mathbb{N}$ and also $\alpha T\mathscr{U} \subseteq \mathscr{U}$ for all $\alpha \in \mathbb{C}$. Hence $p(T)\mathscr{U} = T^*\mathscr{U} \subseteq \mathscr{U}$. The theorem follows from Lemma 5.5. □

Lastly we review the proof in the book. This proof is in essence very similar to the first proof. It brings up a matrix representation of T for easier visualization of what we are doing.

Proof. Assume $T\mathscr{U} \subseteq \mathscr{U}$. By Lemma 5.5 $T^*(\mathscr{U}^\perp) \subseteq \mathscr{U}^\perp$.

Now $\mathscr{V} = \mathscr{U} \oplus \mathscr{U}^\perp$. Let $n = \dim(\mathscr{V})$. Let $\{u_1, \dots, u_m\}$ be an orthonormal basis of $\mathscr{U}$ and let $\{u_{m+1}, \dots, u_n\}$ be an orthonormal basis of $\mathscr{U}^\perp$. Then $\mathscr{B} = \{u_1, \dots, u_n\}$ is an orthonormal basis of $\mathscr{V}$. Since $Tu_j \in \mathscr{U}$ for all $j \in \{1, \dots, m\}$ we have

$$\mathsf{M}_{\mathscr{B}}^{\mathscr{B}}(T) = \begin{array}{c} u_1 \\ \vdots \\ u_m \\ u_{m+1} \\ \vdots \\ u_n \end{array} \left[\begin{array}{ccc|c} \langle Tu_1, u_1 \rangle & \cdots & \langle Tu_m, u_1 \rangle & \\ \vdots & \ddots & \vdots & B \\ \langle Tu_1, u_m \rangle & \cdots & \langle Tu_m, u_m \rangle & \\ \hline & & & \\ & 0 & & C \\ & & & \end{array} \right] \tag{26}$$

$$\quad Tu_1 \quad \cdots \quad Tu_m \quad Tu_{m+1} \quad \cdots \quad Tu_n$$

eq-MBBTis

Here we prepended the basis vectors on the left hand side of the matrix and we appended the images of the basis vectors under T below the matrix to emphasize that an appended vector Tu_k is expended as a linear combination of the basis vectors which are prepended with the coefficients given in the k-th column of the matrix.

For $k \in \{1, \ldots, m\}$ we have $Tu_k = \sum_{j=1}^{m} \langle Tu_k, u_j \rangle u_j$. By the Pythagorean Theorem

$$\|Tu_k\|^2 = \sum_{j=1}^{m} |\langle Tu_k, u_j \rangle|^2 \quad \text{and} \quad \|T^* u_k\|^2 = \sum_{j=1}^{n} |\langle T^* u_k, u_j \rangle|^2.$$

Since T is normal, $\|Tu_k\|^2 = \|T^* u_k\|^2$ for all $k \in \{1, \ldots, m\}$, and therefore $\sum_{k=1}^{m} \|Tu_k\|^2 = \sum_{k=1}^{m} \|T^* u_k\|^2$. Consequently,

$$\begin{aligned}
\sum_{k=1}^{m} \sum_{j=1}^{m} |\langle Tu_k, u_j \rangle|^2 &= \sum_{k=1}^{m} \sum_{j=1}^{n} |\langle T^* u_k, u_j \rangle|^2 \\
&= \sum_{k=1}^{m} \sum_{j=1}^{m} |\langle T^* u_k, u_j \rangle|^2 + \sum_{k=1}^{m} \sum_{j=m+1}^{n} |\langle T^* u_k, u_j \rangle|^2 \\
&= \sum_{k=1}^{m} \sum_{j=1}^{m} |\langle u_k, Tu_j \rangle|^2 + \sum_{k=1}^{m} \sum_{j=m+1}^{n} |\langle T^* u_k, u_j \rangle|^2.
\end{aligned}$$

We have

$$\sum_{k=1}^{m} \sum_{j=1}^{m} |\langle Tu_k, u_j \rangle|^2 = \sum_{k=1}^{m} \sum_{j=1}^{m} |\langle u_k, Tu_j \rangle|^2$$

since these sums consist of identical terms. Hence, the last two displayed equalities yield

$$\sum_{k=1}^{m} \sum_{j=m+1}^{n} |\langle T^* u_k, u_j \rangle|^2 = 0$$

As the last double sum consists of the nonnegative terms we deduce that for all $k \in \{1, \ldots, m\}$ and for all $j \in \{m+1, \ldots, n\}$ we have

$$0 = |\langle T^* u_k, u_j \rangle|^2 = |\langle u_k, Tu_j \rangle|^2 = |\langle Tu_j, u_k \rangle|^2.$$

Hence also $\langle Tu_j, u_k \rangle = 0$ for all $k \in \{1, \ldots, m\}$ and for all $j \in \{m+1, \ldots, n\}$. This proves that $B = 0$ in (26). Therefore, Tu_j is orthogonal to $\mathscr{U}$ for all $j \in \{m+1, \ldots, n\}$, which implies $T(\mathscr{U}^{\perp}) \subseteq \mathscr{U}^{\perp}$. □

Theorem 8.1 and Lemma 5.5 yield the following corollary.

Corollary 8.2. *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$. Let $\langle\cdot,\cdot\rangle$ be a positive definite inner product on $\mathscr{V}$. Let $T \in \mathscr{L}(\mathscr{V})$ be normal and let $\mathscr{U}$ be a subspace of $\mathscr{V}$. The following statements are equivalent:*

(a) $T\mathscr{U} \subseteq \mathscr{U}$.
(b) $T(\mathscr{U}^{\perp}) \subseteq \mathscr{U}^{\perp}$.
(c) $T^{*}\mathscr{U} \subseteq \mathscr{U}$.
(d) $T^{*}(\mathscr{U}^{\perp}) \subseteq \mathscr{U}^{\perp}$.

If any of the for above statements are true, then the following statements are true

(e) $\bigl(T\big|_{\mathscr{U}}\bigr)^{*} = T^{*}\big|_{\mathscr{U}}$.

(f) $\bigl(T\big|_{\mathscr{U}^{\perp}}\bigr)^{*} = T^{*}\big|_{\mathscr{U}^{\perp}}$.

(g) $T\big|_{\mathscr{U}}$ *is a normal operator on* $\mathscr{U}$.

(h) $T\big|_{\mathscr{U}^{\perp}}$ *is a normal operator on* $\mathscr{U}^{\perp}$.

9 Polar Decomposition

There are two distinct subsets of $\mathbb{C}$. Those are the set of nonnegative real numbers, denoted by $\mathbb{R}_{\geq 0}$, and the set of complex numbers of modulus 1, denoted by $\mathbb{T}$. An important tool in complex analysis is the polar representation of a complex number: for every $\alpha \in \mathbb{C}$ there exists $r \in \mathbb{R}_{\geq 0}$ and $u \in \mathbb{T}$ such that $\alpha = r\,u$.

In this section we will prove that an analogous statement holds for operators in $\mathscr{L}(\mathscr{V})$, where $\mathscr{V}$ is a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product. The first step towards proving this analogous result is identifying operators in $\mathscr{L}(\mathscr{V})$ which will play the role of nonnegative real numbers and the role of complex numbers with modulus 1. That is done in the following two definitions.

Definition 9.1. Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle\,\cdot\,,\cdot\,\rangle$. An operator $Q \in \mathscr{L}(\mathscr{V})$ is said to be *nonnegative* if $\langle Qv, v\rangle \geq 0$ for all $v \in \mathscr{V}$.

Note that Axler uses the term "positive" instead of nonnegative. We think that nonnegative is more appropriate, since $0_{\mathscr{L}(\mathscr{V})}$ is a nonnegative operator. There is nothing positive about any zero, we think.

Proposition 9.2. *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle \cdot, \cdot \rangle$ and let $T \in \mathscr{L}(\mathscr{V})$. Then T is nonnegative if and only if T is normal and all its eigenvalues are nonnegative.*

th-sqrt-nno **Theorem 9.3.** *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle \cdot, \cdot \rangle$. Let $Q \in \mathscr{L}(\mathscr{V})$ be a nonnegative operator and let $u_1, \ldots, u_n$ be an orthonormal basis of $\mathscr{V}$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}$ be such that*

$$Qu_j = \lambda_j u_j \quad \text{for all} \quad j \in \{1, \ldots, n\}. \tag{27}$$ eq-sqrt-nno-1

The following statements are equivalent.

i-sqrt-nno-a (a) *$S \in \mathscr{L}(\mathscr{V})$ is a nonnegative operator and $S^2 = Q$.*

i-sqrt-nno-b (b) *For every $\lambda \in \mathbb{R}_{\geq 0}$ we have*

$$\operatorname{nul}(Q - \lambda I) = \operatorname{nul}(S - \sqrt{\lambda} I).$$

i-sqrt-nno-c (c) *For every $v \in \mathscr{V}$ we have*

$$Sv = \sum_{j=1}^{n} \sqrt{\lambda_j} \langle v, u_j \rangle u_j.$$

Proof. (a) $\Rightarrow$ (b). We first prove that $\operatorname{nul} Q = \operatorname{nul} S$. Since $Q = S^2$ we have $\operatorname{nul} S \subseteq \operatorname{nul} Q$. Let $v \in \operatorname{nul} Q$, that is, let $Qv = S^2 v = 0$. Then $\langle S^2 v, v \rangle = 0$. Since S is nonnegative it is self-adjoint. Therefore, $\langle S^2 v, v \rangle = \langle Sv, Sv \rangle = \|Sv\|^2$. Hence, $\|Sv\| = 0$, and thus $Sv = 0$. This proves that $\operatorname{nul} Q \subseteq \operatorname{nul} S$ and (b) is proved for $\lambda = 0$.

Let $\lambda > 0$. Then the operator $S + \sqrt{\lambda} I$ is invertible. To prove this, let $v \in \mathscr{V} \setminus \{0_{\mathscr{V}}\}$ be arbitrary. Then $\|v\| > 0$ and therefore

$$\big\langle (S + \sqrt{\lambda} I)v, v \big\rangle = \langle Sv, v \rangle + \sqrt{\lambda} \langle v, v \rangle \geq \sqrt{\lambda} \|v\|^2 > 0.$$

Thus, $v \neq 0$ implies $(S + \sqrt{\lambda} I)v \neq 0$. This proves the injectivity of $S + \sqrt{\lambda} I$.

To prove $\operatorname{nul}(Q - \lambda I) = \operatorname{nul}(S - \sqrt{\lambda} I)$, let $v \in \mathscr{V}$ be arbitrary and notice that $(Q - \lambda I)v = 0$ if and only if $\big(S^2 - \sqrt{\lambda}^2 I\big)v = 0$, which, in turn, is equivalent to

$$\big(S + \sqrt{\lambda} I\big)\big(S - \sqrt{\lambda} I\big)v = 0.$$

Since $S + \sqrt{\lambda} I$ is injective, the last equality is equivalent to $\big(S - \sqrt{\lambda} I\big)v = 0$. This completes the proof of (b).

(b) ⇒ (c). Let $u_1, \dots, u_n$ be an orthonormal basis of $\mathscr{V}$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq 0}$ be such that (27) holds. For arbitrary $j \in \{1, \dots, n\}$ (27) yields $u_j \in \operatorname{nul}(Q - \lambda_j I)$. By (b), $u_j \in \operatorname{nul}(S - \sqrt{\lambda_j} I)$. Thus

$$Su_j = \sqrt{\lambda_j} u_j \quad \text{for all} \quad j \in \{1, \dots, n\}. \tag{28}$$ eq-sqrt-nno-2

Let $v = \sum_{j=1}^n \langle v, u_j \rangle u_j$ be arbitrary vector in $\mathscr{V}$. Then, the linearity of S and (28) imply the claim in (c).

The implication (c) ⇒ (a) is straightforward. □

The implication (a) ⇒ (c) of Theorem 9.3 yields that for a given nonnegative Q a nonnegative S such that $Q = S^2$ is uniquely determined. The common notation for this unique S is $\sqrt{Q}$.

Definition 9.4. Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle \cdot, \cdot \rangle$. An operator $U \in \mathscr{L}(\mathscr{V})$ is said to be *unitary* if $U^*U = I$.

pr-uop **Proposition 9.5.** *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle \cdot, \cdot \rangle$ and let $T \in \mathscr{L}(\mathscr{V})$. The following statements are equivalent.*

(a) *T is unitary.*
(b) *For all $u, v \in \mathscr{V}$ we have $\langle Tu, Tv \rangle = \langle u, v \rangle$.*
(c) *For all $v \in \mathscr{V}$ we have $\|Tv\| = \|v\|$.*
(d) *T is normal and all its eigenvalues have modulus 1.*

Theorem 9.6 (Polar Decomposition in $\mathscr{L}(\mathscr{V})$)**.** *Let $\mathscr{V}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite inner product $\langle \cdot, \cdot \rangle$. For every $T \in \mathscr{L}(\mathscr{V})$ there exist a unitary operator U in $\mathscr{L}(\mathscr{V})$ and a unique nonnegative $Q \in \mathscr{L}(\mathscr{V})$ such that $T = UQ$; U is unique if and only if T is invertible.*

Proof. First, notice that the operator T^*T is nonnegative: for every $v \in \mathscr{V}$ we have

$$\langle T^*Tv, v \rangle = \langle Tv, Tv \rangle = ||Tv||^2 \geq 0.$$

To prove the uniqueness of Q assume that $T = UQ$ with U unitary and Q nonnegative. Then $Q^* = Q$, $U^* = U^{-1}$ and therefore, $T^*T = Q^*U^*UQ = QU^{-1}UQ = Q^2$. Since Q is nonnegative we have $Q = \sqrt{T^*T}$.

Set $Q = \sqrt{T^*T}$. By Theorem 9.3(b) we have $\operatorname{nul} Q = \operatorname{nul}(T^*T)$. Moreover, we have $\operatorname{nul}(T^*T) = \operatorname{nul} T$. The inclusion $\operatorname{nul} T \subseteq \operatorname{nul}(T^*T)$ is trivial.

For the converse inclusion notice that $v \in \operatorname{nul}(T^*T)$ implies $T^*Tv = 0$, which yields $\langle T^*Tv, v\rangle = 0$ and thus $\langle Tv, Tv\rangle = 0$. Consequently, $\|Tv\| = 0$, that is $Tv = 0$, yielding $v \in \operatorname{nul} T$. So,

$$\operatorname{nul} Q = \operatorname{nul}(T^*T) = \operatorname{nul} T \tag{29}$$ eq-nQ=nT

is proved.

First assume that T is invertible. By (29) and ??, Q is invertible as well. Therefore $T = UQ$ is equivalent to $U = TQ^{-1}$ in this case. Since Q is unique, this proves the uniqueness of U. Set $U = TQ^{-1}$. Since Q is self-adjoint, Q^{-1} is also self-adjoint. Therefore $U^* = Q^{-1}T^*$, yielding $U^*U = Q^{-1}T^*TQ^{-1} = Q^{-1}Q^2Q^{-1} = I$. That is, U is unitary.

Now assume that T is not invertible. Since by (29) we have $\operatorname{nul} Q = \operatorname{nul} T$, the Nullity-Rank Theorem implies that $\dim(\operatorname{ran} Q) = \dim(\operatorname{ran} T)$. Notice that $\operatorname{nul} Q = (\operatorname{ran} Q)^\perp$ since Q is self-adjoint. Since T is not invertible, $\dim(\operatorname{ran} Q) = \dim(\operatorname{ran} T) < \dim \mathscr{V}$, implying that

$$\dim(\operatorname{nul} Q) = \dim\bigl((\operatorname{ran} Q)^\perp\bigr) = \dim\bigl((\operatorname{ran} T)^\perp\bigr) > 0. \tag{30}$$ eq-drQp=drTp

We have two orthogonal decompositions of $\mathscr{V}$:

$$\mathscr{V} = (\operatorname{ran} Q) \oplus (\operatorname{nul} Q) = (\operatorname{ran} T) \oplus \bigl((\operatorname{ran} T)^\perp\bigr).$$

These two orthogonal decompositions are compatibile in the sense that the corresponding components have same dimensions, that is

$$\dim(\operatorname{ran} Q) = \dim(\operatorname{ran} T) \quad \text{and} \quad \dim(\operatorname{nul} Q) = \dim\bigl((\operatorname{ran} T)^\perp\bigr).$$

We will define $U : \mathscr{V} \to \mathscr{V}$ in two steps based on these two orthogonal decompositions. First we define the action of U on $\operatorname{ran} Q$, that is we define the operator $U_r : \operatorname{ran} Q \to \operatorname{ran} T$, then we define an operator $U_n : \operatorname{nul} Q \to (\operatorname{ran} T)^\perp$.

We define $U_r : \operatorname{ran} Q \to \operatorname{ran} T$ in the following way: Let $u \in \operatorname{ran} Q$ be arbitrary and let $x \in \mathscr{V}$ be such that $u = Qx$. Then we set

$$U_r u = Tx.$$

First we need to show that U_r is well defined. Let $x_1, x_2 \in \mathscr{V}$ be such that $u = Qx_1 = Qx_2$. Then, $x_1 - x_2 \in \operatorname{nul} Q$. Since $\operatorname{nul} Q = \operatorname{nul} T$, we thus have $x_1 - x_2 \in \operatorname{nul} T$. Consequently, $Tx_1 = Tx_2$, that is U_r is well defined.

Next we prove that U_r is angle-preserving. Let $u_1, u_2 \in \operatorname{ran} Q$ be arbitrary and let $x_1, x_1 \in \mathscr{V}$ be such that $u_1 = Qx_1$ and $u_2 = Qx_2$ and calculate

$$\langle U_r u_1, U_r u_2\rangle = \bigl\langle U_r(Qx_1), U_r(Qx_2)\bigr\rangle$$

$$
\begin{aligned}
\boxed{\text{by definition of } U_r} &= \langle T x_1, T x_2 \rangle \\
\boxed{\text{by definition of adjoint}} &= \langle T^* T x_1, x_2 \rangle \\
\boxed{\text{by definition of } Q} &= \langle Q^2 x_1, x_2 \rangle \\
\boxed{\text{since } Q \text{ is self-adjoint}} &= \langle Q x_1, Q x_2 \rangle \\
\boxed{\text{by definition of } x_1, x_2} &= \langle u_1, u_2 \rangle
\end{aligned}
$$

Thus $U_r : \operatorname{ran} Q \to \operatorname{ran} T$ is angle-preserving.

Next we define an angle-preserving operator

$$U_n : \operatorname{nul} Q \to (\operatorname{ran} T)^{\perp}.$$

By (30), we can set

$$m = \dim(\operatorname{nul} Q) = \dim\bigl((\operatorname{ran} T)^{\perp}\bigr) > 0.$$

Let $e_1, \ldots, e_m$ be an orthonormal basis on $\operatorname{nul} Q$ and let $f_1, \ldots, f_m$ be an orthonormal basis on $(\operatorname{ran} T)^{\perp}$. For arbitrary $w \in \operatorname{nul} Q$ define

$$U_n w = U_n \left(\sum_{j=1}^{m} \langle w, e_j \rangle e_j \right) := \sum_{j=1}^{m} \langle w, e_j \rangle f_j.$$

Then, for $w_1, w_2 \in \operatorname{nul} Q$ we have

$$
\begin{aligned}
\langle U_n w_1, U_n w_2 \rangle &= \left\langle \sum_{i=1}^{m} \langle w_1, e_i \rangle f_i, \sum_{j=1}^{m} \langle w_2, e_j \rangle f_j \right\rangle \\
&= \sum_{j=1}^{m} \langle w_1, e_j \rangle \overline{\langle w_2, e_j \rangle} \\
&= \langle w_1, w_2 \rangle.
\end{aligned}
$$

Hence U_n is angle-preserving on $(\operatorname{ran} Q)^{\perp}$.

Since the orthomormal bases in the definition of U_n were arbitrary and since $m > 0$, the operator U_n is not unique.

Finally we define $U : \mathscr{V} \to \mathscr{V}$ as a direct sum of U_r and U_n. Recall that

$$\mathscr{V} = (\operatorname{ran} Q) \oplus (\operatorname{nul} Q).$$

Let $v \in \mathscr{V}$ be arbitrary. Then there exist unique $u \in (\operatorname{ran} Q)$ and $w \in (\operatorname{nul} Q)$ such that $v = u + w$. Set

$$U v = U_r u + U_n w.$$

We claim that U is angle-preserving. Let $v_1, v_2 \in \mathscr{V}$ be arbitrary and let $v_i = u_i + w_i$ with $u_i \in (\operatorname{ran} Q)$ and $w_i \in (\operatorname{nul} Q)$ with $i \in \{1,2\}$. Notice that

$$\langle v_1, v_2 \rangle = \langle u_1 + w_1, u_2 + w_2 \rangle = \langle u_1, u_2 \rangle + \langle w_1, w_2 \rangle, \tag{31}$$ eq-pom-1

since u_1, u_2 are orthogonal to w_1, w_2. Similarly

$$\langle U_r u_1 + U_n w_1, U_r u_2 + U_n w_2 \rangle = \langle U_r u_1, U_r u_2 \rangle + \langle U_n w_1, U_n w_2 \rangle, \tag{32}$$ eq-pom-2

since $U_r u_1, U_r u_2 \in (\operatorname{ran} T)$ and $U_n w_1, U_n w_2 \in (\operatorname{ran} T)^{\perp}$. Now we calculate, starting with the definition of U,

$$\begin{aligned} \langle U v_1, U v_2 \rangle &= \langle U_r u_1 + U_n w_1, U_r u_2 + U_n w_2 \rangle \\ \boxed{\text{by (32)}} &= \langle U_r u_1, U_r u_2 \rangle + \langle U_n w_1, U_n w_2 \rangle \\ \boxed{U_r \text{ and } U_n \text{ are angle-preserving}} &= \langle u_1, u_2 \rangle + \langle w_1, w_2 \rangle \\ \boxed{\text{by (31)}} &= \langle v_1, v_2 \rangle. \end{aligned}$$

Hence U is angle-preserving and by Proposition 9.5 we have that U is unitary.

Finally we show that $T = UQ$. Let $v \in \mathscr{V}$ be arbitrary. Then $Qv \in \operatorname{ran} Q$. By definitions of U and U_r we have

$$UQv = U_r Qv = Tv.$$

Thus $T = UQ$, where U is unitary and Q is nonnegative. □

10 Singular Value Decomposition

The following theorem is long. It deals with an arbitrary nonzero operator between finite-dimensional positive definite inner product spaces. Its main parts are (I) and (IV). Part (I) establishes the existence of a Singular Value Decomposition, while in Part (IV), we prove the existence and uniqueness of the Moore-Penrose inverse for such an operator.

th-svd **Theorem 10.1.** *Let $m, n \in \mathbb{N}$. Let $(\mathscr{V}, \langle \cdot\,, \cdot \rangle_{\mathscr{V}})$ and $(\mathscr{W}, \langle \cdot\,, \cdot \rangle_{\mathscr{W}})$ be finite-dimensional positive definite inner product vector spaces over $\mathbb{C}$ such that $m = \dim \mathscr{V}$ and $n = \dim \mathscr{W}$. Let $T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$ be a nonzero operator. Then there exist $r \in \mathbb{N}$ such that $r \leq \min\{m, n\}$, positive scalars $\sigma_1, \ldots, \sigma_r$ and orthonormal bases $\mathscr{B} = \{v_1, \ldots, v_m\}$ of $\mathscr{V}$ and $\mathscr{C} = \{w_1, \ldots, w_n\}$ of $\mathscr{W}$ such that the following statements hold.*

th-svd-i1 (I) *For every* $v \in \mathscr{V}$ *we have*

$$Tv = \sum_{j=1}^{r} \sigma_j \langle v, v_j \rangle_{\mathscr{V}} w_j. \qquad (33)$$ eq-svd

th-svd-i1a (II) *The* $r \times r$ *top left block corner of the* $n \times m$ *matrix* $\mathsf{M}_{\mathscr{C}}^{\mathscr{B}}(T)$ *is the diagonal matrix with the positive diagonal entries* $\sigma_1, \ldots, \sigma_r$ *and all the other entries of* $\mathsf{M}_{\mathscr{C}}^{\mathscr{B}}(T)$ *are equal to* 0. *That is*

$$M_{\mathscr{C}}^{\mathscr{B}}(T) = \begin{array}{r} r\left\{ \vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}} \right. \\ n-r\left\{ \vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}} \right. \end{array} \left[\begin{array}{ccc|ccc} \sigma_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right] \begin{array}{l} n\times m \ \ \text{matrix with} \\ \text{a diagonal } r\times r \\ \text{top left block and with} \\ \text{all other entries zeros.} \end{array}$$

$$\underbrace{\hphantom{\sigma_1 \cdots \sigma_r}}_{r} \quad \underbrace{\hphantom{0 \cdots 0}}_{m-r}$$

Or, in block-matrix notation

$$M_{\mathscr{C}}^{\mathscr{B}}(T) = \left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right], \qquad (n \times m \ \ \text{matrix})$$

where Σ_r *is an* $r \times r$ *diagonal matrix with positive entries* $\sigma_1, \ldots, \sigma_r$ *on the diagonal and the zero matrices of the appropriate sizes.*

th-svd-i2 (III) *For every* $w \in \mathscr{W}$ *we have*

$$T^* w = \sum_{j=1}^{r} \sigma_j \langle w, w_j \rangle_{\mathscr{W}} v_j. \qquad (34)$$ eq-svd*

Equivalently,

$$M_{\mathscr{B}}^{\mathscr{C}}(T^*) = \left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right], \qquad (m \times n \ \ \text{matrix})$$

th-svd-i3 (IV) *Let* $S \in \mathscr{L}(\mathscr{W}, \mathscr{V})$. *The following three statements are equivalent.*

th-svd-i31 (i) S *satisfies the following four equations*

$$TST = T, \quad STS = S, \quad (TS)^* = TS, \quad (ST)^* = ST, \qquad (35)$$ eq-MPiS

th-svd-i32 (ii) *For every* $w \in \mathscr{W}$ *we have*

$$Sw = \sum_{j=1}^{r} \frac{1}{\sigma_j} \langle w, w_j \rangle_{\mathscr{W}} v_j. \tag{36}$$ eq-svdMPi

th-svd-i33 (iii)

$$M_{\mathscr{B}}^{\mathscr{C}}(S) = \left[\begin{array}{c|c} \Sigma_r^{-1} & 0 \\ \hline 0 & 0 \end{array}\right]. \qquad (m \times n \quad \textit{matrix})$$

Proof. (I) Let $T^* \in \mathscr{L}(\mathscr{W}, \mathscr{V})$ be the adjoint of T. Since for all $v \in \mathscr{V}$ we have

$$\langle T^*Tv, v \rangle_{\mathscr{V}} = \langle Tv, Tv \rangle_{\mathscr{W}} \geq 0,$$

the operator $T^*T \in \mathscr{L}(\mathscr{V})$ is nonnegative, and, as such, self-adjoint with the nonnegative eigenvalues $\lambda_1, \ldots, \lambda_n$. We assume that the eigenvalues are ordered in nonincreasing order $\lambda_1 \geq \cdots \geq \lambda_n$. Since $T \neq 0_{\mathscr{L}(\mathscr{V})}$ we have $\lambda_1 > 0$. Set

$$r = \max\bigl\{k \in \{1, \ldots, n\} : \lambda_k > 0\bigr\}. \tag{37}$$ eq-rankr

Thus, for all $k \in \{1, \ldots, n\}$, if $k \leq r$, then $\lambda_k > 0$, and, if $k > r$, then $\lambda_k = 0$. Set

$$\sigma_k = \sqrt{\lambda_k}, \qquad k \in \{1, \ldots, r\}. \tag{38}$$ eq-svs

Since T^*T is self-adjoint, there exists an orthonormal basis $\mathscr{B} = \{v_1, \ldots, v_m\}$ of $\mathscr{V}$ such that

$$\forall k \in \{1, \ldots, n\} \quad T^*Tv_k = \lambda_k v_k. \tag{39}$$ eq-tstep

Recall that

$$\operatorname{nul}(T) = \operatorname{nul}(T^*T) \quad \text{and} \quad \operatorname{ran}(T^*) = \operatorname{ran}(T^*T)$$

It follows from the definition of r in (37) and (39) that

$$\operatorname{nul}(T) = \operatorname{nul}(T^*T) = \operatorname{span}\bigl\{v_k : k \in \{1, \ldots, n\} \wedge k > r\bigr\}.$$

Since T^*T is self-adjoint and since $\mathscr{B}$ is an orthonormal basis of $\mathscr{V}$, (37) and (39) imply

$$\operatorname{ran}(T^*) = \operatorname{ran}(T^*T) = \bigl(\operatorname{nul}(T^*T)\bigr)^{\perp} = \operatorname{span}\bigl\{v_k : k \in \{1, \ldots, n\} \wedge k \leq r\bigr\}.$$

Therefore

$$r = \dim \operatorname{ran}(T^*).$$

Notice that for all $k \in \{1, \dots, r\}$ we have

$$0 < \lambda_k = (\sigma_k)^2 = \lambda_k \langle v_k, v_k \rangle_{\mathscr{V}} = \langle T^* T v_k, v_k \rangle_{\mathscr{V}} = \langle T v_k, T v_k \rangle_{\mathscr{W}} = \|T v_k\|_{\mathscr{W}}^2,$$

and define r unit vectors in $\operatorname{ran}(T) \subseteq \mathscr{W}$ as follows

$$w_k = \frac{1}{\sigma_k} T v_k, \qquad k \in \{1, \dots, r\}.$$

The following calculation shows that the vectors $w_1, \dots, w_r$ are mutually orthogonal. Let $j, k \in \{1, \dots, r\}$ be arbitrary and such that $j \neq k$. Then

$$\langle w_j, w_k \rangle_{\mathscr{W}} = \frac{1}{\sigma_j \sigma_k} \langle T v_j, T v_k \rangle_{\mathscr{W}} = \frac{1}{\sigma_j \sigma_k} \langle T^* T v_j, v_k \rangle_{\mathscr{V}} = \frac{\lambda_j}{\sigma_j \sigma_k} \langle v_j, v_k \rangle_{\mathscr{V}} = 0,$$

since $\mathscr{B}$ is an orhonormal basis for $\mathscr{V}$. Consequently, $w_1, \dots, w_r$ are linearly independent in $\mathscr{W}$. Hence, $r \leq \min\{m, n\}$.

Since

$$r + \dim \operatorname{nul}(T) = m = \dim \mathscr{V}$$

and, by the Nullity-Rank Theorem,

$$\dim \operatorname{nul}(T) + \dim \operatorname{ran}(T) = m = \dim \mathscr{V},$$

we deduce that $r = \dim \operatorname{ran}(T)$. Hence $\{w_1, \dots, w_r\}$ is an orthonormal basis for $\operatorname{ran}(T)$. If $\operatorname{ran}(T)$ is a proper subspace of $\mathscr{W}$, since $\big(\operatorname{ran}(T)\big)^{\perp} = \operatorname{nul}(T^*)$, choosing $w_{r+1}, \dots, w_n$ to be an orthonormal basis for $\operatorname{nul}(T^*)$ we obtain an orthonormal basis $\mathscr{C} = \{w_1, \dots, w_n\}$ for $\mathscr{W}$. Let $v \in \mathscr{V}$ be arbitrary and calculate

$$\begin{aligned}
Tv &= T\left(\sum_{k=1}^{m} \langle v, v_k \rangle_{\mathscr{V}} v_k\right) \\
\boxed{\text{linearity of } T} &= \sum_{k=1}^{m} \langle v, v_k \rangle_{\mathscr{V}} T v_k \\
\boxed{\text{definition of } r} &= \sum_{k=1}^{r} \langle v, v_k \rangle_{\mathscr{V}} T v_k \\
\boxed{\text{definition of } w_k} &= \sum_{k=1}^{r} \langle v, v_k \rangle_{\mathscr{V}} \sigma_k w_k \\
&= \sum_{k=1}^{r} \sigma_k \langle v, v_k \rangle_{\mathscr{V}} w_k.
\end{aligned}$$

(III) Define $S \in \mathscr{L}(\mathscr{W}, \mathscr{V})$ by: For every $w \in \mathscr{W}$ set

$$Sw = \sum_{j=1}^{r} \sigma_j \langle w, w_j \rangle_{\mathscr{W}} v_j .$$

For an arbitrary $v \in \mathscr{V}$ and an arbitrary $w \in \mathscr{W}$ calculate

$$\begin{aligned}
\langle v, Sw \rangle_{\mathscr{V}} &= \left\langle v, \sum_{j=1}^{r} \sigma_j \langle w, w_j \rangle_{\mathscr{W}} v_j \right\rangle_{\mathscr{V}} \\
&= \sum_{j=1}^{r} \sigma_j \overline{\langle w, w_j \rangle_{\mathscr{W}}} \langle v, v_j \rangle_{\mathscr{V}} \\
&= \sum_{j=1}^{r} \sigma_j \langle v, v_j \rangle_{\mathscr{V}} \langle w_j, w \rangle_{\mathscr{W}} \\
&= \left\langle \sum_{j=1}^{r} \sigma_j \langle v, v_j \rangle_{\mathscr{V}} w_j, w \right\rangle_{\mathscr{W}} \\
&= \langle Tv, w \rangle_{\mathscr{W}} .
\end{aligned}$$

Since $v \in \mathscr{V}$ and $w \in \mathscr{W}$ were arbitrary, the preceding calculation proves that $S = T^*$. (citation)

(IV) The equivalence (ii)⇔(iii) follows from the definition of the matrices Σ_r and $M_{\mathscr{C}}^{\mathscr{B}}(S)$.

To prove (ii)⇒(i), assume (ii). (This is proof of the existence of the Moore-Penrose inverse.) Then, (34) and (36) imply that $\operatorname{ran}(S) = \operatorname{ran}(T^*)$ and $\operatorname{nul}(S) = \operatorname{nul}(T^*)$. Further, (33) and (36) yield

$$TS = P_{\operatorname{ran}(T)} = P_{\operatorname{nul}(S)^\perp} \quad \text{and} \quad ST = P_{\operatorname{ran}(S)} = P_{\operatorname{nul}(T)^\perp} .$$

Consequently, TS and ST are self-adjoint (citation), and, since $P_{\operatorname{ran}(T)} T = T$ and $P_{\operatorname{ran}(S)} S = S$, we deduce that $TST = T$ and $STS = S$. Thus (ii)⇒(i).

To prove (i)⇒(iii), assume (i). (This is proof of the uniqueness of the Moore-Penrose inverse.) Let

$$M_{\mathscr{B}}^{\mathscr{C}}(S) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \qquad (m \times n \ \ \text{matrix})$$

where A is an $r \times r$ matrix, B is $r \times (n-r)$ matrix, C is $(m-r) \times r$ matrix, and D is $(m-r) \times (n-r)$ matrix. We proved in (II)

$$M_{\mathscr{C}}^{\mathscr{B}}(T) = \left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right], \qquad (n \times m \ \ \text{matrix})$$

with Σ_r being an $r\times r$ diagonal matrix with positive entries on the diagonal and the zeros of appropriate sizes. Then

$$M_{\mathscr{C}}^{\mathscr{C}}(TS) = \left[\begin{array}{c|c} \Sigma_r A & \Sigma_r B \\ \hline 0 & 0 \end{array}\right], \qquad (n \times n \ \text{ matrix})$$

and

$$M_{\mathscr{B}}^{\mathscr{B}}(ST) = \left[\begin{array}{c|c} A\Sigma_r & 0 \\ \hline C\Sigma_r & 0 \end{array}\right]. \qquad (m \times m \ \text{ matrix})$$

Since TS and ST are self-adjoint, we deduce that $\Sigma_r B = 0$ and $C\Sigma_r = 0$. Consequently, $B = 0$ and $C = 0$ as Σ_r is invertible. Since $TST = T$, the operator TS acts as an identity on $\operatorname{ran}(T)$. Therefore $\Sigma_r A = I_r$. Hence $A = \Sigma_r^{-1}$. Hence,

$$M_{\mathscr{B}}^{\mathscr{C}}(S) = \left[\begin{array}{c|c} \Sigma_r^{-1} & 0 \\ \hline 0 & D \end{array}\right].$$

Now the equality $S = STS$ yields

$$\begin{aligned}
\left[\begin{array}{c|c} \Sigma_r^{-1} & 0 \\ \hline 0 & D \end{array}\right] &= M_{\mathscr{B}}^{\mathscr{C}}(S) \\
&= M_{\mathscr{B}}^{\mathscr{C}}(STS) \\
&= M_{\mathscr{B}}^{\mathscr{C}}(S) M_{\mathscr{C}}^{\mathscr{B}}(T) M_{\mathscr{B}}^{\mathscr{C}}(S) \\
&= \left[\begin{array}{c|c} \Sigma_r^{-1} & 0 \\ \hline 0 & D \end{array}\right] \left[\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} \Sigma_r^{-1} & 0 \\ \hline 0 & D \end{array}\right] \\
&= \left[\begin{array}{c|c} \Sigma_r^{-1} & 0 \\ \hline 0 & D \end{array}\right] \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \\
&= \left[\begin{array}{c|c} \Sigma_r^{-1} & 0 \\ \hline 0 & 0 \end{array}\right].
\end{aligned}$$

Hence, $D = 0$, and consequently,

$$M_{\mathscr{B}}^{\mathscr{C}}(S) = \left[\begin{array}{c|c} \Sigma_r^{-1} & 0 \\ \hline 0 & 0 \end{array}\right].$$

This proves (i)$\Rightarrow$(iii). Since we proved

$$\text{(i)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)}$$

proof of (IV) is complete. ☐

The values $\sigma_1, \ldots, \sigma_r$ from Theorem 10.1, which are in fact the square-roots of the positive eigenvalues of T^*T, are called *singular values* of T. Equality (33) or the matrix in (II) is called a *singular value decomposition* of T.

For $T \in \mathscr{L}(\mathscr{V}, \mathscr{W})$, the unique operator $T^+ \in \mathscr{L}(\mathscr{W}, \mathscr{V})$ that satisfies the equalities

$$TT^+T = T, \quad T^+TT^+ = T^+, \quad (TT^+)^* = TT^+, \quad (T^+T)^* = T^+T. \tag{40}$$

eq-MPi

is called the *Moore-Penrose inverse* of T,

11 Problems

Exercise 11.1. Let $(\mathscr{V}, \langle \cdot, \cdot \rangle)$ be a positive definite inner product space and let $\mathscr{U}$ be a subspace of $\mathscr{V}$. Prove that $\bigl((\mathscr{U}^{\perp})^{\perp}\bigr)^{\perp} = \mathscr{U}^{\perp}$.