Riesz Basis for Indefinite Sturm-Liouville Problems

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Consider boundary eigenvalue problem

$$\ell(f) = \sum_{j=0}^{n} (-1)^j \Big(p_j f^{(j)}\Big)^{(j)} = \lambda r f \quad \text{on} \quad [-1, 1]$$

with boundary conditions of the form

$$\begin{aligned} \mathbf{L}\,\mathbf{b}(f) &= \mathbf{0} \\ \mathbf{M}\mathbf{b}(f) &= \lambda \mathbf{N}\mathbf{b}(f) \end{aligned}$$

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- $p_n > 0$ and r changes sign (indefinite)

To which extend the spectral theory of this indefinite problem parallels the spectral theory for the case $r > 0$?

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- There exists a Riesz basis of the form domain of A consisting of the root vectors of A.

What is the form domain $\mathcal{F}(A)$ of A?

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$\mathcal{F}(A)$ is a subspace of $L_{2,r}(-1,1) \oplus \mathbb{C}^m$

It consists of vectors of the form

$$\begin{bmatrix} f \\ \mathbf{N}_e \mathbf{b}_e(f) \\ \mathbf{v} \end{bmatrix}$$

$$f, f', \ldots, f^{(n-1)} \in AC[-1,1]$$
$$\int_{-1}^{1} p_n |f^{(n)}|^2 < +\infty$$
$$\mathbf{D}_e \mathbf{b}_e(f) = \mathbf{0}$$

These results parallel the corresponding results for the case $r = 1$ in

M. G. Krein,

The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. II. (Russian)

Mat. Sbornik N.S. 21(63), (1947), 365–404.

Does there exist a Riesz basis of

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This is much harder question!

A. I. Parfënov in
Sibirsk. Mat. Zh. 44 (2003), 810–819

considered a special case

$$-f'' = \lambda\, r\, f \quad \text{on} \quad [-1, 1],$$

with the Dirichlet boundary conditions

$$f(-1) = f(1) = 0,$$

where r is an odd function in $L_1(-1,1)$ such that

$$r > 0 \quad \text{on} \quad [0, 1].$$

Parfënov's answer:

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there exist $c \geq 1$ and $\gamma > 0$ such that

$$\int_0^{tx} r(\xi) d\xi \leq c\, t^{\gamma} \int_0^{x} r(\xi) d\xi$$

for all $t, x \in (0, 1]$.

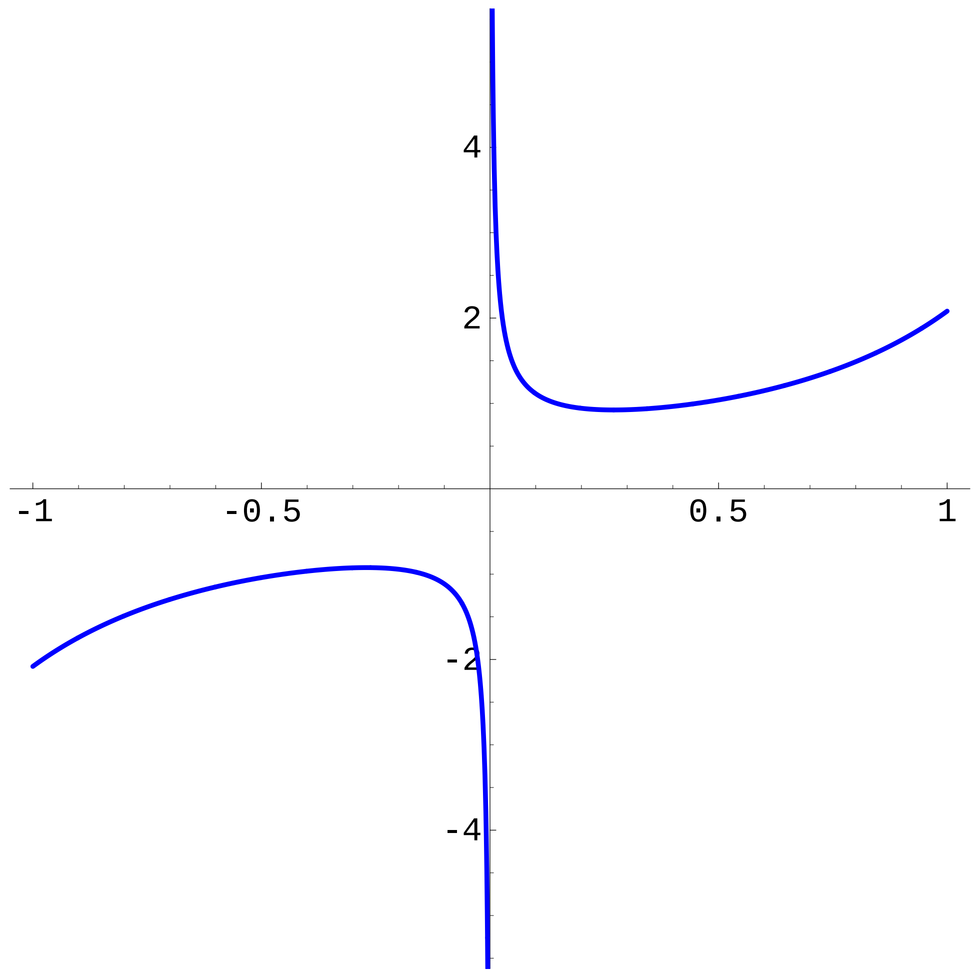
Let $u > 1$. The function

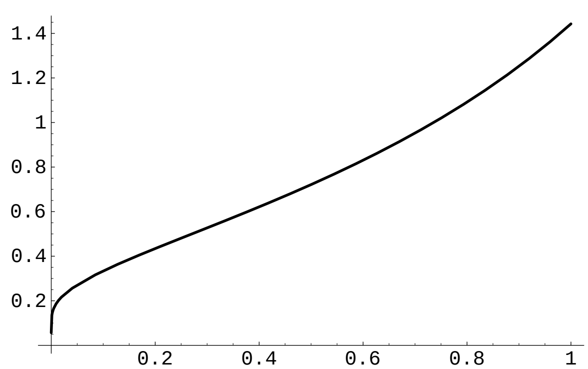
$$r_u(x) = \frac{1}{x(\ln u - \ln |x|)^2}, \quad x \in [-1, 1] \setminus \{0\}$$

does not satisfy the Parfënov condition, since

$$\int_0^x r_u(\xi)\, d\xi = \frac{1}{\ln u - \ln x}, \quad x \in [0, 1]$$

The function r_u





$$\int_0^x r_u(\xi)\, d\xi$$

No set of eigenfunctions of the problem

$$-f''(x) = \lambda \frac{1}{x(\ln u - \ln|x|)^2} f(x),$$

$$f(-1) = f(1) = 0,$$

forms a Riesz basis of $L_{2,r_u}(-1,1)$.

Does Parfënov's criteria hold true for all self-adjoint boundary conditions?

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No!

There exists an odd function r, $r > 0$ on $[0, 1]$, satisfying the Parfënov condition and such that no set of eigenfunctions of the eigenvalue problem

$$-f'' = \lambda\, r\, f,$$

$$f(-1) + f(1) = 0,$$

$$f'(-1) + f'(1) = 0,$$

is a Riesz basis of $L_{2,r}(-1, 1)$.

Consider a more general problem

$$-f'' + qf = \lambda\, r\, f \qquad \text{on} \qquad [-1, 1],$$

with

general self-adjoint boundary conditions

and

a function r which is not necessarily odd

(for simplicity I assume that $x\, r(x) > 0$)

Theorem.

Let $\mathcal{F}(A)$ denote the form domain of A.

There exists a Riesz basis of

$$L_{2,r}(-1,1) \oplus \mathbb{C}^m_\Delta$$

consisting of root vectors of A

if and only if

there exists a bounded, uniformly positive operator W in the Krein space $L_{2,r}(-1,1) \oplus \mathbb{C}^m_\Delta$ such that

$$W\mathcal{F}(A) \subset \mathcal{F}(A).$$

Let

$a, b \in [-1, 1]$, $\quad h_a,\ h_b \subset [-1, 1]$

half-neighborhoods of a and b

We say that h_a and h_b are

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- $\epsilon > 0$,
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  $\alpha : [0, \epsilon] \to h_a$ and $\beta : [0, \epsilon] \to h_b$,
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such that

- $\alpha(0) = a$ and $\beta(0) = b$,
- $|r(\alpha(t))|\ \rho(t) = |r(\beta(t))|$,
- $|\alpha'| \neq |\beta'| \rho(0)$

Let $a \in [-1, 1]$ and
let $h_a \subset [-1, 1]$ be a half-neighborhood of a.

If there exists $\nu > -1$ and $g_1 \in C^1(h_a)$
such that

$$r(x) = |x - a|^\nu g_1(x) \quad \text{and} \quad g_1(x) \neq 0, \quad x \in h_a,$$

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h_a is smoothly r-connected to h_b if also

$$r(x) = |x - b|^\nu g_2(x) \quad \text{and} \quad g_2(x) \neq 0, \quad x \in h_b,$$

where $g_2 \in C^1(h_b)$

Condition at 0.

Denote by 0_- a generic left and by 0_+ a generic right half-neighborhood of 0. At least one of the four pairs of half-neighborhoods

$$(0_-, 0_-), \quad (0_-, 0_+), \quad (0_+, 0_-), \quad (0_+, 0_+),$$

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is smoothly r-connected.

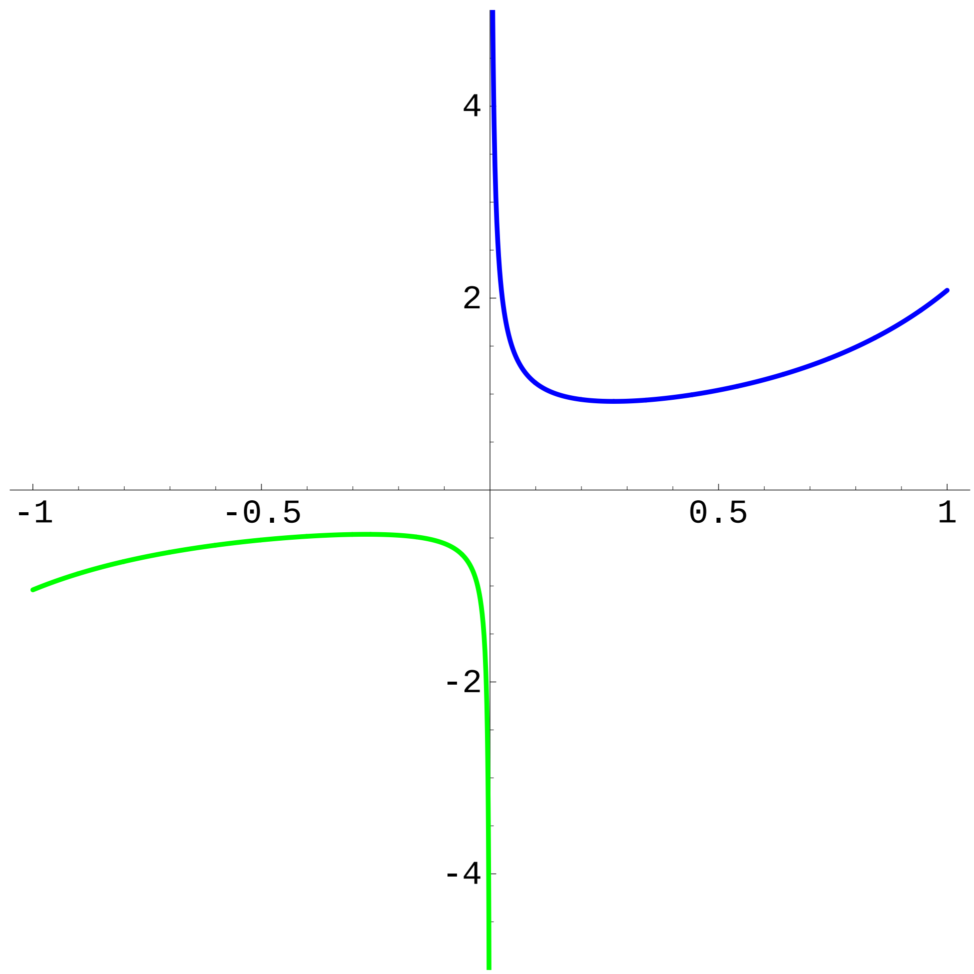
"Slightly non-odd functions"
Let $g \in L_1(0,1),\ g > 0,$ e.g.,

$$g(x) = \frac{1}{x(\ln u - \ln x)^2},\ x \in [0,1]$$

Let $0 < v \neq 1$. Put

$$r(x) = \begin{cases} g(x), & x \in [0,1] \\ -vg(-x), & x \in [-1,0) \end{cases}$$

“Slightly non-odd function”



Condition at -1.

A right half-neighborhood of -1 is smoothly r-connected to itself.

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- For λ-independent boundary conditions either of the above conditions is sufficient for Riesz-basis property of A. ("one-sided" condition)

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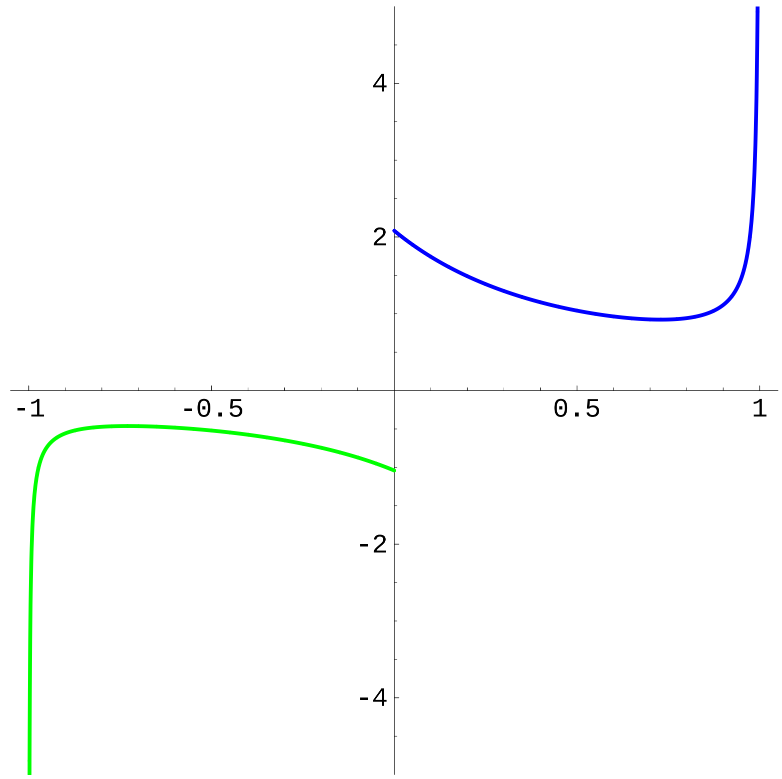
- For λ-independent boundary conditions <u>either</u> of the above conditions is sufficient for Riesz-basis property of A. ("one-sided" condition)

- If one boundary condition is λ-dependent our method, in some cases, does not allow a free choice of the condition. For example

For the problem

$$\begin{aligned} -f'' &= \lambda\, r\, f \\ f'(1) &= 0 \\ -f'(-1) &= \lambda f(-1) \end{aligned}$$

our method requires Condition at -1 for the proof of the Riesz-basis property.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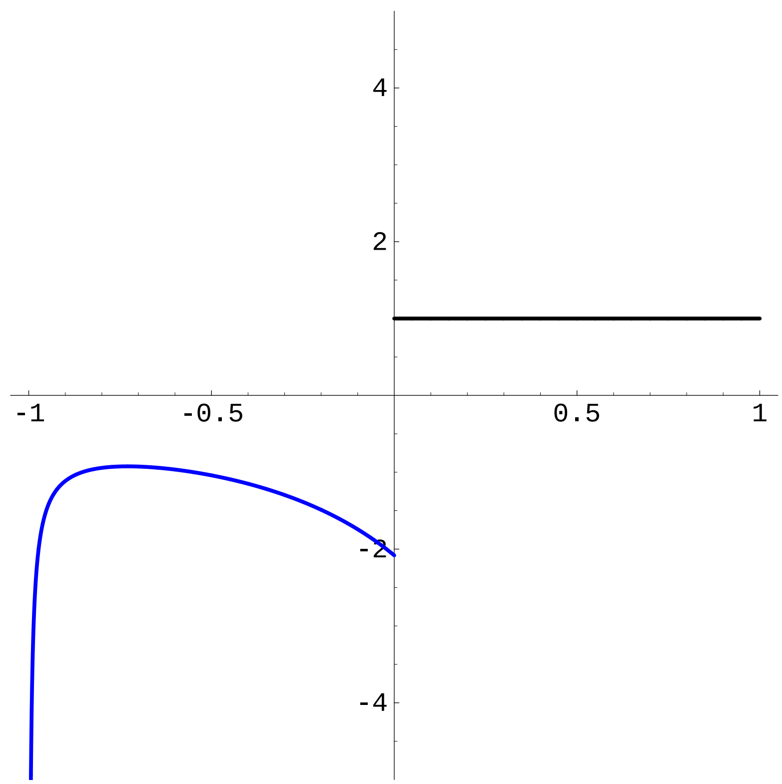
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If both boundary conditions are λ-dependent, then our method, in some cases, requires all the above conditions and

Mixed Condition at ± 1.

There are
two smooth r-connections between
a right half-neighborhood of -1 and
a left half-neighborhood of 1
with the connection parameters α'_j, β'_j and $\rho_j(0)$, $j = 1, 2$, such that

$$\begin{vmatrix} |\alpha'_1| & |\alpha'_2| \\ |\beta'_1|\rho_1(0) & |\beta'_2|\rho_2(0) \end{vmatrix} \neq 0.$$

Example

For the problem

$$
\begin{aligned}
-f'' &= \lambda\, r\, f \\
f'(1) &= \lambda f(-1) \\
-f'(-1) &= \lambda f(1)
\end{aligned}
$$

our method requires all the stated conditions to be satisfied to prove the Riesz-basis property.

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$$

our method requires all the stated conditions to be satisfied to prove the Riesz-basis property.

Here $\Delta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$
\mathcal{F}(A) = \left\{ \begin{bmatrix} f \\ f(-1) \\ f(1) \end{bmatrix} \in \begin{array}{c} L_{2,r} \\ \oplus \\ \mathbb{C}^2_{\boldsymbol{\Delta}} \end{array} \;:\; f \in H^1[-1,1] \right\}
$$

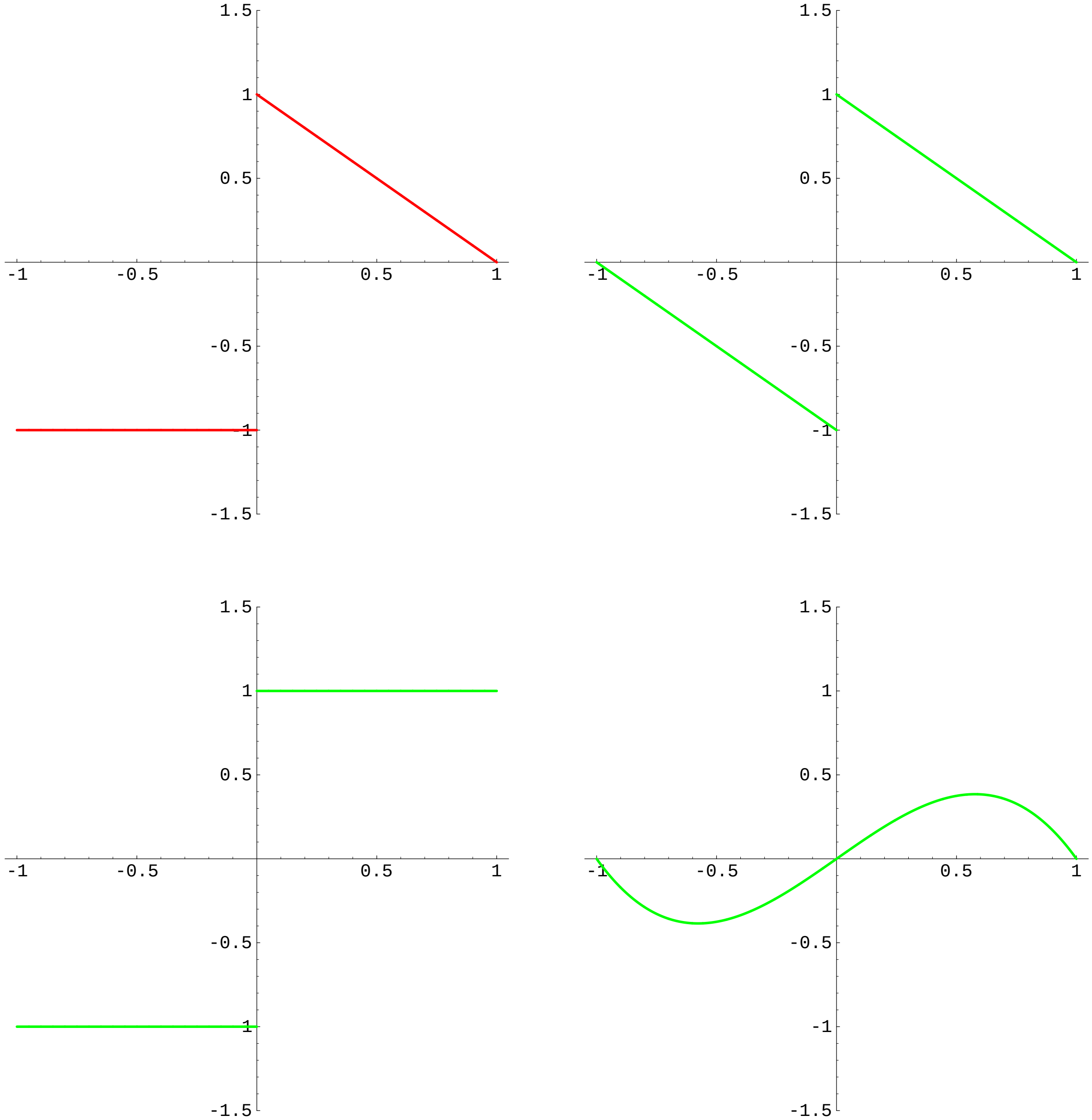
$$
\begin{aligned}
-f'' &= \lambda\, r\, f \\
f'(1) &= \lambda f(-1) \\
-f'(-1) &= \lambda f(1)
\end{aligned}
$$

The function

$$
r(x) = \begin{cases} -1, & x \in [-1, 0) \\ 1 - x, & x \in [0, 1] \end{cases}
$$

does not satisfy Mixed Condition at ± 1.

Our method fails to prove the Riesz-basis property for the above problem.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